The Libyan Academy

The Basic Sciences School

Department of Mathematics

A study On Some Properties Of Intuitionistic Fuzzy Interior Ideals Of Γ-Semigroups

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Submitted by:

Susan Mustafa Ali Atman

Supervised by:

Dr. Ibrahim Abd-Allah Tentush

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مقدمة من الطالبة:

سوسن مصطفي علي عثمان

إشراف الدكتور:

إبراهيم عبد الله تنتوش

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Dedication

I would like to dedicate kind regards and respects to my honest father and mother who showed me the right way of my life and to all members of my family and my friends.

I sincerely dedicate this modest work.

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In the name of Allah, the most beneficent and the most merciful, all praises and thanks are due to Allah for giving me the ability to write with project. First, I would like to thank my supervisor Dr. Ibrahim Abd-Allah Tentoush who being the most generous with his time providing valuable guidance.

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Introduction

After the introduction of fuzzy sets by Zadeh[16], reconsideration of the concept of classical mathematics began. On the other hand, because of the importance of group theory in mathematics, as well as its many areas of application, the notion of fuzzy subgroups was defined by Rosenfeld [4] and its structure was investigated. Das characterized fuzzy subgroups by their level subgroups in [24]. Nobuaki Kuroki [1, 22] is the pioneer of fuzzy ideal theory of semigroups. The idea of fuzzy subgroup was also introduced by Kuroki[1, 22]. In 2007,Uckun, Ozturk and Jun [21] introduced the notions of intuitionistic fuzzy ideals in Γ-semigroups. Motivated by Kuroki [1, 22], Uckun et.al. [1],Sardar et al.[1, 30] have initiated the study of Γ-semigroups in terms of fuzzy sets. We study the notion of intuitionistic fuzzy interior ideals and intuitionistic fuzzy characteristic interior ideals of Γ-semigroup. Finally, In this study we will see characterization of simple Γ-semigroup in terms of intuitionistic fuzzy interior ideal.

This study will be organized as follows:

Chapter I : Γ-semigroups

1.1-semigroup

1.2-(m, n)-ideals in semigroups:

1.3- Γ-semigroup

Chapter II :Intuitionistic fuzzy sets:

2.1-Fuzzy set

2.2-Intuitionistic fuzzy sets

Chapter III :Intuitionistic fuzzy interior ideals of Γ-semigroups:

3.1-Intuitionistic fuzzy subsemigroups

3.2-Intuitionistic fuzzy interior ideals of Γ-semigroup

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Chapter One

Γ-Semigroups

Chapter two

Intuitionistic Fuzzy sets

Chapter three

Intuituinistic fuzzy Interior Ideals Of Γ-Semigroups

Abstract

This study is devoted to the concept of intuitionistic fuzzy interior ideals of Γ-semigroup, by using Atanssov's idea and investigate some of their important properties.We will see a characterization of a simple Γ-semigroup in terms of intuitionistic fuzzy interior ideals.

This chapter is devoted to the concepts that we need in the following chapters. The materials in this chapter are taken from the following references: [2], [3], [5], [6], [7], [9], [10], [16], [17], [18], [19], [20], [22], [23], [26], [27], [28],[29], [30], [33], [34], [35], [36], [37], [38].

1.1 Relations:

Some describe or define mathematics as the study of relations. Since a relation is a set of ordered pairs, we get our first glimpse of the fundamental importance of the concept of an ordered pair.

Definition 1.1.1:

If A and B are any non-empty sets, then the product (or Cartesian product) $A \times B$ is defined by:

$$A \times B = \{(a, b): a \in A, b \in B\}.$$

Definition 1.1.2:

A binary relation or simply a relation R from a set A into a set B is a subset of $A \times B$, and we write that aRb or $(a, b) \in R$.

Definition 1.1.3:

Let R be a relation on the set S. Then

(i) A relation R is called reflexive if $(a, a) \in R$ for all $a \in S$.

(ii) A relation R is called irreflexive if $(a, a) \notin R$ for some $a \in S$.

(iii) A relation R is called anti-reflexive if $(a, a) \notin R$ for all $a \in S$.

(iv) A relation R is called symmetric if $(a, b) \in R$ implies $(b, a) \in R$ for any $a, b \in S$.

(v) A relation R is called anti-symmetric if $(a, b) \in R$ and $(b, a) \in R$ implies $a = b$ for all $a, b \in S$.

(vi) A relation R is called transitive if $(a, b) \in R$ and $(b, c) \in R$ implies $(a, c) \in R$ for all $a, b, c \in S$.

Definition 1.1.4:

A relation R on a set S is an equivalence relation, if R is reflexive, symmetric, and transitive.

Definition 1.1.5 Let R be an equivalence relation on a set S. Then equivalence class of any element $a \in S$ denoted by $[a]$, or R_a is the set $\{x \in S: xRa\}$.

Theorem 1.1.1:

Let R be an equivalence relation on a set S. Then (i) For all $a \in S, [a] \neq \emptyset$.

(ii) If $b \in [a]$ then $[b] = [a]$.

(iii) For all $a, b \in S$, either $[a] = [b]$ or $[a] \cap [b] = \emptyset$.

(iv) $S = \bigcup_{a \in S}[a]$, is the union of all equivalence classes with respect to R.

Chapter One: Γ-Semigroups

Proposition 1.1.1: Any equivalence relation on S gives a partition for S and any partition for S gives equivalence relation on S.

Proposition 1.1.2: The intersection of any family of an equivalence relations on S is an equivalence relation on S.

Example 1.1.1:

Let $S = \{2, 3, 4, 5\}$, and R_1, R_2, R_3 are equivalence relations on S, such that:

R_1={(2, 2),(3, 3), (4, 4), (5, 5),(2, 5), (5,2)} R_2={(2, 2), (3, 3), (4, 4), (5, 5), (2, 5), (5, 2),(2, 4), (4, 2), (4, 5), (5,4)}, R_3= {(2, 2), (3, 3), (4, 4), (5, 5), (2, 5), (5, 2), (2, 3), (3, 2), (3, 5), (5, 3)},

Then, $\bigcap_{i=1}^{3} R_i = \{(2,2),(3,3),(4,4),(5,5),(2,5),(5,2)\}$. And $\bigcup_{i=1}^{3} R_i = \{(2,2),(2,3),(2,4),(2,5),(3,2),(3,3),(3,5),(4,2),(4,4),(4,5),(5,2),(5,3),(5,4),(5,5)\}$

It is clear that $\bigcap_{i=1}^{3} R_i$ equivalence relation on S. Now, (4, 2), (2, 3)$\in \bigcup_{i=1}^{3} R_i$, but (4, 3) $\notin \bigcup_{i=1}^{3} R_i$.

Hence the union of any family of an equivalence relations is not necessary an equivalence relation.

1.2 Semigroups:

Definition 1.2.1: A binary operation $*$ on a set S is a function: $*: S \times S \longrightarrow S$. For any $a, b \in S$, we shall write $a * b$ *instead of* $*(a, b)$.

Definition 1.2.2:

(i) A binary operation $*$ on a set S is commutative (Abelian) if :

$$a * b = b * a \text{ for all } a, b \in S.$$

(ii) A binary operation $*$ on a set S is associative if :

$$(a * b) * c = a * (b * c) \text{ for all } a, b, c \in S.$$

Definition 1.2.3:

A groupoid is an order pair $(S, *)$, where S is a non-empty set and $*$ is closed binary operation on S.

Definition 1.2.4:

A semigroup is an order pair $(S, *)$, where S is a non-empty set and $*$ is an associative binary operation on S.

Example 1.2.1:

The order pair $(\mathbb{Z}^+, +)$ is a semigroup, where $\mathbb{Z}^+$ is the set of positive integers.

Example 1.2.2:

Let $S = \{1, 2, 3\}$ be the non-empty set with the binary operation defined by the multiplication as follows:

$*$	1	2	3
1	1	1	1
2	1	2	2
3	1	2	3

Table 1.2.1

Then S is a semigroup.

Definition 1.2.5:

A semigroup $(S, *)$ is commutative if $*$ is commutative, i.e.,

$a * b = b * a$ for all $a, b \in S$.

A semigroup $(S, *)$ which is not commutative is called non-commutative.

Example 1.2.3:

$(\mathbb{N}, +)$ is a commutative semigroup under usual addition, where $\mathbb{N}$ is the set of natural numbers.

Definition 1.2.6:

Let $(S, *)$ be a semigroup. If the number of elements in S is finite, we say that S is a finite semigroup or a semigroup of finite order. Otherwise, S is called infinite semigroup.

Definition 1.2.7:

Let $(S, *)$ be a semigroup. If there exist an element $e \in S$ such that $a * e = e * a \ \forall a \in S$, then " e " is called the identity element of S.

Definition 1.2.8:
A semigroup with identity is called a monoid.

Example 1.2.4:

Let $M_{2\times 2} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in Z \right\}$. Then $M_{2\times 2}$ is non-commutative monoid under multiplication of matrices.

Definition 1.2.9:

If a semigroup S with at least two elements contains an element 0 such that, for all x in S,

$$0x = x0 = 0$$

We say that 0 is a zero element (or just a zero) of S, and that S is a semigroup with zero.

Example 1.2.5:

$(\mathbb{R}, .)$ is a semigroup where $\mathbb{R}$ the set of real numbers and " **.** " the usual multiplication. Then 0 is a zero element of S.

Definition 1.2.10:

Let S be a semigroup. An element $a \in S$ is said to be an idempotent in S if $a^2 = a$.

Example 1.2.6: Z_{12} under multiplication modulo 12 has 4 as an idempotent.

Definition 1.2.11

Let $(S, *)$ be a semigroup. A non–empty subset A of S is said to be a subsemigroup of S if A is itself semigroup under the same operation.

Definition 1.2.12:

For any two subsets A,B of a semigroup S, define their product by:

$$AB = \{ab: a \in A, b \in B\}.$$

Remark 1.2.1:

Let $(S, *)$ be a semigroup. Anon-empty subset A of S is said to be a subsemigroup if $A^2 \subseteq A$, where $(A, *)$ is a semigroup.

Example 1.2.7: The set $A = \{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in \mathbb{Z}\}$ is a subsemigroup of $M_{2\times2}(\mathbb{Z})$, under addition or multiplication of matrices.

Example 1.2.8:

$A = \{0, 2, 4, 8\}$ is a subsemigroup of Z_{12} under multiplication modulo 12.

Example 1.2.9:

Consider the additive semigroup $S = (\mathbb{Q}, +)$ where $\mathbb{Q}$ is the set of rationals. Now, $(\mathbb{N},+)$ is subsemigroup of S where $\mathbb{N}$ is the set of naturals.

Theorem 1.2.1:

Let S be a semigroup, let $\{ A_i : i \in I \}$ be a family of subsemigroups of S, and let

$$T = \bigcap_{i \in I} A_i$$

With $T \neq \emptyset$. Then T is a subsemigroup of S.

<u>Definition 1.2.13:</u>

A non-empty subset A of a semigroup S is called

(i)A right ideal of S, if $AS \subseteq A$.

(ii)A left ideal of S, if $SA \subseteq A$.

<u>Example 1.2.10:</u>

Let S be the set of all integers and $M_2(S)$, the set of all 2×2 matrices over S. Then $M_2(S)$ is a semigroup under the usual multiplication.

Let $A = \{ \begin{bmatrix} x & 0 \\ y & 0 \end{bmatrix} : x, y \in S \}$, and $B = \{ \begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix} : x, y \in S \}$.

Then A is a left ideal of $M_2(S)$, and B is right ideal of $M_2(S)$.

<u>Definition 1.2.14:</u>

A non-empty subset A of a semigroup S is called ideal of S, if it is both left and right ideal.

<u>Example 1.2.11:</u>

.	1	2	3

Let $S = \{1, 2, 3\}$ be the semigroup defined by the following multiplication table: Table 1.2.2

1	1	1	1
2	1	2	2
3	1	2	3

Let $A = \{1, 2\}$. Then A is a subsemigroup of S.

And since $SA = \{1, 2\} = A$, and $AS = \{1, 2\} = A$. Then A is a right and left ideal of S.

Thus A is an ideal of S.

Example1.2.12:

$A = \{0, 4\}$, and $B = \{0, 2, 4, 6\}$ are ideals of $\mathbb{Z}_8$ under multiplication modulo 8.

Lemma 1.2.1:

If A is a (left or right) ideal of a semigroup S. Then A is a subsemigroup of S.

Proof:

If $SA \subseteq A$ or, $AS \subseteq A$, then $AA \subseteq A$, since $A \subseteq S$.

Thus A is a subsemigroup of S. ■

Lemma 1.2.2:

If A and B are a left (right) ideals of a semigroup S with $A \cap B \neq \emptyset$, then $A \cap B$ is a left (right) ideal of S.

Proof: Let $a \in A \cap B$ and $s \in S$. Then $sa \in A$, and $sa \in B$, since A and B are ideals, and so $sa \in A \cap B$. Hence $(A \cap B)S \subseteq (A \cap B)$. ■

Definition 1.2.15:

A subsemigroup A of a semigroup S is called an interior ideal of S if $SAS \subseteq A$.

Definition 1.2.16: Let S be a semigroup and m, n non-negative integers. We say that S is (m, n)-regular if it satisfies:

$$(\forall\, a \in S)(\exists\, x \in S)(a = a^m x a^n).$$

(a^0 is defined as an operator element, so $a^0 x = x a^0 = x$).

Remark 1.2.2:

If $m = n = 1$ then a semigroup S is called regular.

Definition 1.2.17:

A semigroup S is called completely regular if it satisfies:

$$(\forall\, a \in S)(\exists x \in S)(a = axa\,, ax = xa).$$

Example 1.2.13:

Consider the semigroup $(\mathbb{Z}, +)$ where $\mathbb{Z}$ is the set of integers and "+" the usual addition, then S is a regular semigroup and is completely regular semigroup.

Definition 1.2.18:

A semigroup S is called intra-regular if it satisfied:

$(\forall\, a \in S)(\exists x, y \in S)(a = xa^2y)$.

Example 1.2.14:

Let S be a kroneker semigroup, that is, S has a zero 0, and $xy = \begin{cases} x & if \quad x = y \\ 0 & otherwise \end{cases}$

And assume that $|S| > 2$. Since $x^2 = x$ for all $x \in S$, S is regular, intra-regular and (m, n)-regular.

1.3 (m, n)-ideals in semigroups:

Definition 1.3.1:

Let S be a semigroup. A subsemigroup A of S is called (m, n)-ideal of S, if A satisfies the condition

$A^mSA^n \subseteq A$ ……………………………………………………..(1)

Where m, n are non-negative integers. (Here A^0 is defined as an operator element, so that $A^0S = SA^0 = S$). By is a proper (m, n)-ideal of S we mean that A is a proper subset of S. The concept of (m, n)-ideal is a generalization of one-sided (left or right) ideals in a semigroups.

Remark 1.3.1:

If $m = n = 1$ then A is called bi-ideal or $(1, 1)$-ideal of S.

Example 1.3.1:

$S = [0, 1]$ is a semigroup under the usual multiplication, and let $A = [0, \frac{1}{2}]$. Then A is subsemigroup of S, and since $ASA = [0, \frac{1}{4}] \subseteq A$. Thus A is a bi-ideal of S.

Example 1.3.2:

$A = 2\mathbb{N}$ is a subsemigroup of $\mathbb{N}$ under the usual multiplication is a bi-ideal of $\mathbb{N}$ ($A\mathbb{N}A = 4\mathbb{N} \subseteq 2\mathbb{N} = A$).

Example 1.3.3:

Let S ={0, 1, 2 } be the semigroup defined by the following multiplication table: Table 1.3.1

.	0	1	2
0	0	0	0
1	0	1	1
2	0	2	2

Then sets {0}, {0,1}, {0,2} and {0, 1, 2} are bi-ideals of S.

Example 1.3.4:

Let S ={2, 4, 6, 8, …} be a semigroup under the usual addition "+" and let A ={4, 6, 8, …}. Note $(A, +)$ is a subsemigroup of S.

$SA \subseteq A$, $A(SA) = ASA \subseteq A$. Then A is a bi-ideal of S.

Theorem 1.3.1: Let $\{A_i : i \in I\}$ be a collection of bi-ideals of a semigroup S. Then $\bigcap_{i\in I} A_i$ is a bi-ideal of S if $\bigcap_{i\in I} A_i \neq \emptyset$.

Proof:

Let $A = \bigcap_{i\in I} A_i$. Now to show that A is also bi-ideal. Let $x \in ASA$, then $x = asb$ for some $a, b \in A$ and $s \in S$. Thus $x = asa \in A_i$ for any arbitrary $i \in I$ as $a, b \in A_i$.

So since A_i is bi-ideal, then $x \in ASA \subseteq A_i$ and therefore $x \in A_i$. Since i was chosen arbitrary so $x \in A_i$ for all $i \in I$, then $x \in A$ and hence $ASA \subseteq A$.

Thus $A = \bigcap_{i\in I} A_i$ is a bi-ideal of S. ■

Theorem 1.3.2:

Let S be a semigroup, T be a subsemigroup of S and let A be an (m, n)-ideal of S. Then the intersection $A \cap T$ is an (m, n)-ideal of the semigroup T.

Proof :

The intersection $A \cap T$ evidently is a subsemigroup of S.

$$(A \cap T)^m T (A \cap T)^n \subseteq A^m S A^n \subseteq A \text{……………….}(1)$$

Because of A is an (m, n)-ideal of S. And

$$(A \cap T)^m T (A \cap T)^n \subseteq T^m T T^n \subseteq T \text{....................}(2)$$

Therefore (1)and (2) imply

$$(A \cap T)^m T (A \cap T)^n \subseteq A \cap T,$$

That is the intersection $A \cap T$ is an (m, n)-ideal of T. ■

Theorem 1.3.3:

Let S be a semigroup, A be an (m, n)-ideal of S and let B be a subset of S satisfying either $AB \subseteq A$ or $BA \subseteq A$. Then the products AB and BA are (m, n)-ideals of S (m, n are positive integers).

Theorem 1.3.4:

Let S be an arbitrary semigroup. If L is a left ideal and R is a right ideal of S, then the product RL is a bi-ideal of S.

Proof :

Since

$$(RL)(RL) \subseteq RL$$

The product RL is a subsemigroup of S. On the other hand

$$(RL)S(RL) \subseteq RSL \subseteq RL,$$

that is, the product RL is a bi-ideal of S.

If S is a regular semigroup, with, $a \in aSa$ for each element a in S, then the converse of theorem (1.3.4) also holds. ■

Theorem 1.3.5:

The product of a bi-ideal B and a non-empty subset A of a semigroup S is also a bi-ideal of S.

Theorem 1.3.6:

A subset A of a regular semigroup S is a bi-ideal of S if and only if S contains a left ideal L and a right ideal R, such that $A = RL$.

Definition 1.3.2:

Let S be a semigroup. An ideal P of S is called prime ideal of S if for ideals A and B of S, $AB \subseteq P$ implies that $A \subseteq P$ or $B \subseteq P$.

Definition 1.3.3:

Let S be a semigroup. An ideal P of S is called semiprime ideal of S if for ideal A of S, $A^2 \subseteq P$ implies that $A \subseteq P$.

Definition 1.3.4:

A semigroup S is called right (left) simple if S itself is the only right (left) ideal of S.

Definition 1.3.5: A semigroup S is called simple if it contains no proper ideal.

Example 1.3.5:

In Example (1.2.10), that S is not simple.

1.4 Γ- Semigroups:

Definition 1.4.1:

Let S and Γ be two non-empty sets. Then S is called a one sided Γ-semigroup if there exists a mapping from $S\times \Gamma \times S$ to S, satisfying:

(i) $x\gamma y \in S \forall x, y \in S, \forall \gamma \in \Gamma$.

(ii) $(x\beta y)\gamma z = x\beta(y\gamma z) \forall x, y, z \in S, \forall \beta, \gamma \in \Gamma$.

Example 1.4.1:

Let Γ be the set of positive odd numbers, and let S be the set of natural numbers. Define $x\gamma y = x+\gamma+y \ \forall x, y \in S, \forall \gamma \in \Gamma$. Then, $\mathbb{N}$ is a Γ-semigroup.

Example 1.4.2:

If $S = M_{m\times n}(\mathbb{R})$ is the set of $m\times n$ matrices and Γ is a set of some $n\times m$ matrices over the field of real numbers, then we can define $A_{m,n}\alpha_{n,m}B_{m,n}$ such that

$$(A_{m,n}\alpha_{n,m}B_{m,n})\beta_{n,m}C_{m,n} = A_{m,n}\alpha_{n,m}(B_{m,n}\beta_{n,m}C_{m,n}),$$

Where $A_{m,n}, B_{m,n}, C_{m,n} \in S$ and $\alpha_{n,m}, \beta_{n,m} \in \Gamma$. An algebraic system that satisfying the associativity property of the above type is a Γ-semigroup.

Definition 1.4.2:

Let S and Γ be two non-empty sets. Then S is called a both sided Γ-semigroup if there exist mappings from $S \times \Gamma \times S$ to S, written as $(a, \alpha, b) \to a\alpha b$, and from $\Gamma \times S \times \Gamma$ to Γ, written as $(\alpha, a, \beta) \to \alpha a\beta$, satisfying the following associative laws:

$(a\alpha b)\beta c = a(\alpha b\beta)c = a\alpha(b\beta c)$ and $\alpha(a\beta b)\gamma = (\alpha a\beta)b\gamma = \alpha a(\beta b\gamma)$, for all a, b, $c \in S$ and for all $\alpha, \beta, \gamma \in \Gamma$.

Example 1.4.3:

Let S be the set of all non-positive integers and Γ be the set of all non-positive even integers. Then S is a Γ-semigroup where $a\alpha b$ and $\alpha a\beta$ for all $a, b \in S$ and $\alpha, \beta \in \Gamma$ denote usual multiplication of integers.

Example 1.4.4:

Let $S = \{3n + 2 : n$ is positive integer$\}$ and $\Gamma = \{3n + 1 : n$ is a positive integer$\}$. Then S is a Γ-semigroup where $a\alpha b = a + \alpha + b$ and $\alpha a\beta = \alpha + a + \beta$ for all $a, b \in S$ and $\alpha, \beta \in \Gamma$, $+$ is the usual addition of integers.

Definition 1.4.3:

A mapping $\boldsymbol{\Phi}$ from a one sided Γ-semigroup S to another a one sided Γ-semigroup T is called a homomorphism if

$$\Phi(x\gamma y) = \Phi(x)\gamma\, \Phi(y) \text{ for all } x, y \in S \text{ and } \gamma \in \Gamma.$$

Definition 1.4.4:

A Γ-semigroup S is called a commutative a both sided Γ-semigroup if $a\gamma b = b\gamma a$ for all $a, b \in S$ and $\gamma \in \Gamma$.

Definition 1.4.5:

Let S be a Γ-semigroup. A non-empty subset A of a Γ-semigroup S is said to be a Γ-subsemigroup of S if $A\Gamma A \subseteq A$, (i.e., if $a\gamma b \in A$ for all $a, b \in A$ and $\gamma \in \Gamma$).

Example 1.4.5:

Let S= [0, 1] and Γ ={$\frac{1}{n}$:n is a positive integer}. Then S is a Γ-semigroup under the usual multiplication, let $A = [0,\frac{1}{2}]$ be a non-empty subset of S, and since $a\gamma b \in A$ for all $a, b \in A$ and $\gamma \in \Gamma$. Then A is a Γ-subsemigroup of S.

Definition 1.4.6:

A subsemigroup A of a Γ-semigroup S is called an interior ideal of S if $S\Gamma A\Gamma S \subseteq A$.

Definition 1.4.7:

Let S be a Γ-semigroup. A non-empty subset A of S is called

(i)left ideal of S if $S\Gamma A \subseteq A$.

(ii)right ideal of S if $A\Gamma S \subseteq A$.

Definition 1.4.8:

Let S be a Γ-semigroup. A non-empty subset A of S is called an ideal of S if it is both a left and right ideal of S.

Definition 1.4.9:

Let S be a Γ-semigroup. A Γ-subsemigroup A of S is called a bi-ideal of S if $A\Gamma S\Gamma A \subseteq A$.

Example 1.4.6:

Let $S = (0, 1)$ and $\Gamma = \{1\}$. Then S is a Γ-semigroup under the usual multiplication. Let $\mathbb{N}$ be the set of all positive integers. For $n \in \mathbb{N}$, let $A_n = (0, \frac{1}{n})$. Then A_n is a bi-ideal of S for all $n \in \mathbb{N}$.

Theorem 1.4.1:

Let S be a Γ-semigroup and A_i a bi-ideal of S for all $i \in I$. If $\bigcap_{i\in I} A_i \neq \emptyset$, then $\bigcap_{i\in I} A_i$ is a bi-ideal of S.

Proof:

Let S be a Γ-semigroup and A_i a bi-ideal of S for all $i \in I$.

Assume that $\bigcap_{i\in I} A_i \neq \emptyset$. Let $a, b \in \bigcap_{i\in I} A_i$, $m \in S$ and $\alpha, \beta \in \Gamma$.

Then $a, b \in A_i$ for all $i \in I$. Since A_i a bi-ideal of S for all $i \in I$, then $a\alpha b \in A_i$ and $a\alpha m\beta b \in A_i \Gamma S \Gamma A_i \subseteq A_i$ for all $i \in I$. Therefore $a\alpha b \in \bigcap_{i\in I} A_i$ and $a\alpha m\beta b \in \bigcap_{i\in I} A_i$ for all $i \in I$.

Hence $\bigcap_{i\in I} A_i$ is a bi-ideal of S. ■

Definition 1.4.10:

Let S be a Γ-semigroup. Then an ideal P of S is said to be

(i)prime if for ideals A, B of S, $A\Gamma B \subseteq P$ implies that $A \subseteq P$ or $B \subseteq P$.
(ii) semiprime if for an ideal A of S, $A\Gamma A \subseteq P$ implies that $A \subseteq P$.

Definition 1.4.11:

A Γ-semigroup S is called left simple (right simple) if it contains no proper left ideals (right ideals).

Definition 1.4.12:

A Γ-semigroup S is called simple if it contains no proper ideals, that is, for any ideal A of S, we have $A = S$.

Example 1.4.7:

Let S be the set of all integers of the form $4n+1$ and Γ be the set of all integers of the form $4n+3$ where n is any integer. Then, S is a simple Γ-semigroup.

Theorem 1.4.2

Let P be an ideal of a Γ-semigroup S. Then the following are equivalent:

(i)P is prime (semiprime).

(ii) for $x, y \in S, x\Gamma S\Gamma y \subseteq P \Rightarrow x \in P$ or $y \in P$ (resp. $x\Gamma S\Gamma x \subseteq P \Rightarrow x \in P$).

(iii) for $x, y \in S, x\Gamma y \subseteq P \Rightarrow x \in P$ or $y \in P$ (resp. $x\Gamma x \subseteq P \Rightarrow x \in P$).

Proof:

(i)$\Rightarrow$ (ii):

Suppose that (i) holds and $x\Gamma S\Gamma y \subseteq P$. Then $< x >\Gamma< x > \Gamma < y > \Gamma < y >\subseteq P$.

Since $<x > \Gamma < x >\subseteq P$, $<y > \Gamma < y >\subseteq P$ are ideals of S, so by (i) we have either $<x > \Gamma < x >\subseteq P$ or $<y > \Gamma < y >\subseteq P$.

By repeated uses of (i) we get $x \in< x >\subseteq P$ or $y \in< y >\subseteq P$.

(ii)$\Rightarrow$ (iii):

Suppose that (ii) holds and $x\Gamma y \subseteq P$. Then $x\Gamma s\Gamma y \subseteq P$ as $x\Gamma s\Gamma y \subseteq x\Gamma y$.

Hence by (ii), $x \in P$ or $y \in P$.

(iii)$\Rightarrow$ (i):

Suppose that (iii) holds. Let for two ideals A, B of S, $A\Gamma B \subseteq P$. If possible, suppose $A \nsubseteq P$ or $B \nsubseteq P$. Then $x \in A$ and $y \in B$ such that

$x \notin P$ and $y \notin P$. This implies that $x\Gamma y \subseteq P$ with $x \notin P, y \notin P$. This is a contradiction to (iii). Hence either $A \subseteq P$ or $B \subseteq P$. Consequently P is a prime ideal of S.

■ And similarly for P is semiprime of S.

Definition 1.4.14:

An element $x \in S$ is called regular in S if $x \in x\Gamma S\Gamma x$.

Definition 1.4.15:

A Γ-semigroup S is called regular if every element is regular.

2.1 Fuzzy set:

In 1965 Zadeh [10] mathematically formulated the fuzzy subset concept. He defined fuzzy subset of a non-empty set as a collection of objects with grade of membership in a continuum, with each object being assigned a value between 0 and 1 by a membership function. Fuzzy set theory was guided by the assumption that classical sets were not natural, appropriate or useful notions in describing then real life problems, because every object encountered in this real physical world carries some degree of fuzziness. Further the concept of grade of membership is not a probabilistic concept.

The materials in this chapter is taken from the following references:

[8], [11], [15], [16], [25], [31], [32], [39].

Definition 2.1.1:

Any function μ from any non-empty set S into the closed interval [0, 1] is called a fuzzy subset in S.

The set of all fuzzy subsets of S is called fuzzy power set of S and is dented by $\mathcal{FP}(S)$=[0, 1].

<u>**Remark2.1.1**</u>: If S is the universal set and $A \subseteq S$ then the function $\mathfrak{X}_A$:S→[0,1] such that

$$\mathfrak{X}_A(x) = \begin{cases} 1 & if \quad x \in A \\ 0 & if \quad x \notin A' \end{cases}$$

is called the characteristic function of A which is a fuzzy subset of S.

<u>**Example 2.1.1:**</u>

Let S= {1, 2, 3, 4, 5}, A= {1, 2, 3} is a subset of S.

The mapping μ: S→[0, 1], such that $\mathfrak{X}_A(1) = 1, = 1, (\mathfrak{X}_A(x)2) = 1, \mathfrak{X}_A(3) = 1,$ $\mathfrak{X}_A(4) = 0$, $\mathfrak{X}_A(5) = 0$, is a fuzzy subset of S.

<u>**Example 2.1.2:**</u>

Let S be a set of real numbers, and μ considerably larger than 8 subset of S, can be defined as a fuzzy set. Where $\mu(x) =$

$$\begin{cases} 0 & x \leq 8 \\ (1 + (x - 8)^{-2})^{-1} & x > 8 \end{cases}$$

<u>**Remark 2.1.2:**</u>

.1. If the universal set S is discrete, the representation of a fuzzy subset μ of S can be expressed by

$$\mu = \sum \mu(x_i)/x_i.$$

Where $\sum$ indicate the union of the elements $\mu(x_i)$ which is the membership degree of x_i .

.2. If the universal set S is not discrete then, the fuzzy set is expressed by

$$\mu = \int_S \mu(x)/x$$

Where $\int$ indicate the union of the elements of μ .

Example 2.1.3:

Let S={3, 4, 5, 6, 7, 8, 9}, and μ " integers close to 6 " subset of S, where $\mu = \{(x, \mu(x)) \mid \mu(x) = (1 + (x - 6)^2)^{-1}\}$ can be defined a fuzzy set as follows:

$$\mu = \{ (3, 0.1), (4, 0.2), (5, 0.5), (6, 1), (7, 0.5), (8, 0.2), (9, 0.1)\}.$$

Can be written as :

$$\mu = 0.1/3 + 0.2/4 + 0.5/5 + 1/6 + 0.5/7 + 0.2/8 + 0.1/9 \quad .$$

Definition2.1.2:

A fuzzy set μ in S is empty if and if $\mu(x) = 0 \ \forall x \in S$.

Definition 2.1.3:

A fuzzy set μ in S is universal if and if $\mu(x) = 1 \ \forall x \in S$.

Definition 2.1.4:

Let $\mu \in \mathcal{FP}(S)$. Then the set $\{\mu(x) \mid x \in S\}$ is called the image of μ and is denoted by $\mu(x)$ or $\mathrm{im}(\mu)$.

Definition 2.1.5:

The support of a fuzzy set μ, is a subset of the universal set S defined as:

$$supp(\mu) = \{x \in S :\ \mu(x) > 0\}.$$

Example 2.1.4:

Let $S=\{0, 1, 2, 3, 4, 5, 6\}$. And μ be a fuzzy subset of S, defined by:

$$\mu=\{(0, 0.3), (1, 0.6), (2, 1), (3, 0), (4, 0.7), (5, 0.8), (6, 0)\}$$

Then $supp(\mu) = \{0, 1, 2, 4, 5\}$.

Remark 2.1.3:

μ is called a finite fuzzy subset if $supp(\mu)$ is a finite set and an infinite fuzzy subset otherwise.

Definition 2.1.6:

Let μ be a fuzzy subset of a set S. For $t \in [0,1]$ the set $\mu_t = \{x \in S \mid \mu(x) \geq t\}$ is called a *t*-level or " *t*-cut " subset of the fuzzy subset μ.

$\mu_{s.t} = \{x \in S \mid \mu(x) > t\}$ is called " strong *t*-level set " or " strong *t*-cut" .

Example 2.1.5:

Let S= { 0,1, 2, 3, 4 } and let μ ={(0, 0.5), (1, 0), (2, 0.3), (3, 0.4), (4, 0.9)},

$\mu_{0.4} = \{x \in S: \mu(x) \geq 0.4\} = \{0, 3, 4\}$.

$\mu_1 = \{x \in S: \mu(x) \geq 1\}$.

$\mu_1 = \emptyset$.

$\mu_{s,0.4} = \{0, 4\}$.

Definition 2.1.7:

Let $Y \subseteq S$ and $a \in (0, 1]$, we define $a_Y \in \mathcal{FP}(S)$ as follows:

$$a_Y(x) = \begin{cases} a & for \quad x \in Y \\ 0 & for \quad x \in S\backslash Y \end{cases}$$

If Y is a singleton, say {y}. Then $a_{\{y\}}$ is called fuzzy point or " fuzzy singleton" and denoted by a_y .

Example2.1.6:

Let S= {a, b, c, d} and $t \in [0, 1]$ define $\mu : S \rightarrow [0, 1]$ by

$$\mu(x) = \begin{cases} t & if \quad x = a \\ 0 & if \quad other\ wise \end{cases}$$

Then t_a is fuzzy point.

Definition 2.1.8:

Let $\mu, v \in \mathcal{FP}(S)$. Then

i. $\mu \subseteq v$ " or $v \supseteq \mu$ " if $\mu(x) \leq v(x)\ \forall\ x \in S$.

ii. $\mu \subset v$ if $\mu(x) \leq v(x)\ \forall\, x \in S$ and there exits at least one $x \in S$ such that $\mu(x) < v(x)$.

iii. $\mu = v$ if $\mu(x) = v(x)\ \ \forall\, x \in S$.

Definition 2.1.9:

A fuzzy subset μ of a set S is said to be normal if $sup_{x \in S}\, \mu(x) = 1$. A fuzzy subset μ of a set S is said to be normalized if there exists $x \in S$ such that $\mu(x) = 1$.

Example 2.1.7:

Let $S = [0, 1]$ and define a fuzzy subset $\mu : S \rightarrow [0, 1]$ by $\mu(x) = x$ for all $x \in S$. Then μ is normal fuzzy subset also it's normalized because there exist $x \in [0,1]$, such that $\mu(x) = 1$.

Definition 2.1.10:

The complement of a fuzzy set, μ^C is defined by

$$\mu^{\mathrm{C}} = 1 - \mu.$$

Example 2.1.8:

Let $S = \{1, 2, 3, 4, \ldots, 12\}$, and

$\mu = \{(1, 0.3), (2. 0.4), (3, 0.6),(4, 0.7), (5, 1), (6, 0.2), (7, 0.8), (8, 0), (9, 0), (10, 0), (11, 0), (12, 0)\}$.

Then $\mu^C = \{(1, 0.7), (2, 0.6), (3, 0.4), (4, 0.3), (5, 0), (6, 0.8), (7, 0.2), (8, 1), (9, 1), (10, 1), (11, 1), (12, 1)\}$.

Remark 2.1.4:

We use the notation $\vee$ for supermum and $\wedge$ for infimum.

Definition 2.1.11:

The union of two fuzzy subsets μ and v of S is a fuzzy set λ. Whose membership function is related to those of μ and v by:

$$\lambda = (\mu \cup v) = \sup\{\mu, v\} = \mu \vee v .$$

Definition 2.1.12:

The intersection of two fuzzy subset μ and v of S is fuzzy set λ. Whose membership function is related to those of μ and v by:

$$\lambda = (\mu \cap v) = inf\{\mu, v\} = \mu \wedge v$$

Example 2.1.9:

Let $S = \{\alpha, \beta, \gamma, \delta\}$ and let

$$\mu = \{(\alpha, 0.1), (\beta, 0.5), (\gamma, 0.6), (\delta, 0.3)\}.$$

$$v = \{(\alpha, 0.3), (\beta, 0.4), (\gamma, 0.6), (\delta, 0.7)\}.$$

Then

$$(\mu \cup v) = \{(\alpha, 0.3), (\beta, 0.5), (\gamma, 0.6), (\delta, 0.7)\}.$$

And

$$(\mu \cap v) = \{(\alpha, 0.1), (\beta, 0.4), (\gamma, 0.6), (\delta, 0.3)\}.$$

And

$$(\mu \cup v)_t = \begin{cases} S & 0.1 \leq t \leq 0.3 \\ \{\beta, \gamma, \delta\} & 0.3 < t \leq 0.5 \\ \{\gamma, \delta\} & 0.5 < t \leq 0.6 \\ \{\delta\} & 0.6 < t \leq 0.7 \\ \emptyset & 0.7 < t \leq 0.9 \end{cases}$$

$$(\mu \cap v)_t = \begin{cases} S & 0 \leq t \leq 0.1 \\ \{\beta, \gamma, \delta\} & 0.1 < t \leq 0.3 \\ \{\beta, \gamma\} & 0.3 < t \leq 0.4 \\ \{\gamma\} & 0.4 < t \leq 0.6 \\ \emptyset & 0.6 < t \leq 0.9 \end{cases}$$

Chapter Two: Intuitionistic Fuzzy Sets

Definition 2.1.13:

Let $\{A_i\}_{i \in I}$ be any arbitrary family of fuzzy subsets of a set. Then

i. $(\cup_i A_i)(x) = \vee_i A_i(x) = sup_i A_i(x)$.

ii. $(\cap_i A_i)(x) = \wedge_i A_i(x) = inf_i A_i(x)$.

Example 2.1.10:

Let S be the set of all real numbers, and let A be the set of all real numbers which are close to 3, and let the membership function of A be defined by:

$$A(x) = \frac{1}{1+(x-3)^2}\ , x \in S.$$

The set of all real numbers which are close to 5, may have a membership function B defined by:

$$B(x) = \frac{1}{1+(x-5)^2}, x \in S.$$

Then, $(A \cup B)(x) = \begin{cases} \frac{1}{1+(x-3)^2} & ,x \leq 4 \\ \frac{1}{1+(x-5)^2} & ,x \geq 4 \end{cases}$

$$(A \cap B)(x) = \begin{cases} \frac{1}{1+(x-5)^2} & ,x \leq 4 \\ \frac{1}{1+(x-3)^2} & ,x \geq 4 \end{cases}$$

Definition 2.1.14:

If μ and v are two fuzzy sets, the relative complement of μ with respect to v, denoted by $\mu - v$, defined by $(\mu - v)(x) = \mu(x) - v(x)$ provided that $\mu \geq v$.

Definition 2.1.15:

The sum of two fuzzy sets μ and v is defined by

$(\mu + v)(x) = \mu(x) + v(x)$ provided that the sum

$\mu(x) + v(x) \leq 1$ for all x.

Example 2.1.11:

Let $S = \{ 1, 2, 3, 4\}$ μ and v be two fuzzy subset of S. $\mu(x) = \begin{cases} 0.7 & if & x = 1, 2 \\ 0.5 & if & x = 3, 4 \end{cases}$

And

$$v(x) = \begin{cases} 0.3 & if & x = 1, 2 \\ 0.4 & if & x = 3, 4 \end{cases}$$

Then,

$$(\mu+v)(x)=\begin{cases}1 & if \quad x=1,2\\ 0.9 & if \quad x=3,4\end{cases}$$

$$(\mu-v)(x)=\begin{cases}0.4 & if \quad x=1,2\\ 0.1 & if \quad x=3,4\end{cases}$$

Definition 2.1.16:

The product of μ and v is defined as $(\mu v)(x)=\mu(x)v(x)$ for all $x \in S$.

Definition 2.1.17:

The probabilistic sum of μ and v is defined as: $(\mu \oplus v)(x)=\mu(x)+v(x)-\mu(x)v(x)$.

Example 2.1.12:

Let $S=\{1,2,3,4\}$ and μ, v be fuzzy subset of S.

$$\mu(x)=\begin{cases}0.3 & if \quad x=1,2\\ 0.4 & if \quad x=3,4\end{cases}, \quad v(x)=\begin{cases}0.5 & if \quad x=1,2\\ 0.3 & if \quad x=3,4\end{cases} \quad (\mu v)(x)=\begin{cases}0.15 & if \quad x=1,2\\ 0.12 & if \quad x=3,4\end{cases}$$

$$(\mu \oplus v)(x)=(\mu+v)-\mu v=\begin{cases}0.65 & if \quad x=1,2\\ 0.58 & if \quad x=3,4\end{cases}$$

Definition 2.1.18:

Let f be function from X into Y, and let $\mu \in \mathcal{F}P(X)$ and $v \in \mathcal{F}P(Y)$, define the fuzzy subset $f(\mu) \in \mathcal{FP}(Y)$ and $f^{-1}(v) \in \mathcal{FP}(X)$ by $\forall\, y \in Y$.

$$f(\mu) = \begin{cases} \sup\{\mu(x) | x \in X, f(x) = y\} & if \quad f^{-1}(y) \neq \emptyset \\ 0 & other\ wise \end{cases}$$

And $\forall\, x \in X$, $f(\mu)$ is called img of μ under f.

$f^{-1}(v)(x) = v\big(f(x)\big), f^{-1}(v)$ is called the pre-imge “or inverse imge” of v under f.

Theorem 2.1.1:

Let f be a function from X into Y and g a function from Y into Z , then the following assertions hold

1) For all $\mu_i \in \mathcal{FP}(X), i \in I, f(\bigcup_{i\in I} \mu_i) = \bigcup_{i \in I} f(\mu_i)$ and so $\mu_1 \subseteq \mu_2 \Rightarrow f(\mu_1) \subseteq f(\mu_2)\ \forall\ \mu_1, \mu_2 \in \mathcal{FP}(X)$.

2) For all $v_j \in \mathcal{FP}(Y), j \in J$ where J is a non empty index set $f^{-1}(\bigcup_{j\in J} v_i) = \bigcup_{j\in J} f^{-1}(v_j)$.

$f^{-1}\big(\bigcap_{j\in J} v_j\big) = \bigcap_{j\in J} f^{-1}\big(v_j\big)$.

And there for $v_1 \subseteq v_2 \Rightarrow f(v_1) \subseteq f(v_2) \forall\ v_1, v_2 \in \mathcal{FP}(Y)$.

3) $f^{-1}(f(\mu)) \supseteq \mu\ \forall \mu \in \mathcal{FP}(X)$

In particular, if f is an injection,

Then $f^{-1}\big(f(\mu)\big) = \mu \ \forall \ \mu \in \mathcal{F}P(X)$.

4) $f\big(f^{-1}(v)\big) \subseteq v \ \forall \ v \in \mathcal{F}P(Y)$, in particular, if f is a surjection, then $f\big(f^{-1}(v)\big) = v \ \forall \ v \in \mathcal{F}P(Y)$.

5) $f(\mu) \subseteq v \Leftrightarrow \mu \subseteq f^{-1}(v) \ \forall \ \mu \in \mathcal{F}P(X)$ and $\forall \ v \ \in \mathcal{F}P(Y)$.

6) $g(f(\mu)) = (g \circ f)(\mu) \ \forall \ \mu \in \mathcal{F}P(X)$.

Remark 2.1.5:

If A is a fuzzy set, it is not necessary that $A \cup A^c = S$ or $A \cap A^c = \emptyset$. For example let $S = \{\alpha, \beta, \delta\}$, and let A be a fuzzy set in S given by:

$$A = \{(\alpha, 1), (\beta, 0.4), (\delta, 0.7)\}.$$

$A^c = \{(\alpha, 0), (\beta, \ 0.6), (\delta, 0.3 \ )\}$. $A \cup A^c = \{(\alpha, 1), (\beta, 0.6), (\delta, 0.7)\} \neq$ the universal set.

$A \cap A^c = \{(\alpha, 0), (\beta, 0.4), (\delta, 0.3)\} \ \neq$ the fuzzy empty set

Definition 2.1.19:

Let μ_1 be a fuzzy subset of a set X_1 and μ_2 be a fuzzy subset of a set X_2, then the fuzzy union of the fuzzy sets μ_1 and μ_2 is defined as a function.

$\mu_1 \cup \mu_2 : X_1 \cup X_2 \to [0,1]$ given by $(\mu_1 \cup \mu_2)(x) =$

$$\begin{cases} \max(\mu_1(x), \mu_2(x)) & if \quad x \in X_1 \cap X_2 \\ \mu_1(x) & if \quad x \in X_1 \setminus X_1 \cap X_2 \\ \mu_2(x) & if \quad x \in X_2 \setminus X_1 \cap X_2 \end{cases}$$

Example 2.1.13:

Let X_1={0, 1, 2, 3, 4} and X_2={1, 3, 5, 7, 9} be two sets. Define $\mu_1 : X_1 \to [0,1]$

$$\mu_1(x) = \begin{cases} 0 & if \quad x = 0, 1 \\ 0.2 & if \quad x = 2 \\ 0.7 & if \quad x = 3, 4 \end{cases}$$

$$\mu_2(x) = \begin{cases} 1 & if \quad x = 1, 3 \\ 0.5 & if \quad x = 5 \\ 0.3 & if \quad x = 7, 9 \end{cases}$$

Then the fuzzy union of the fuzzy sets μ_1 and μ_2

$$(\mu_1 \cup \mu_2)(x) = \begin{cases} 0 & if \quad x = 0 \\ 0.2 & if \quad x = 2 \\ 1 & if \quad x = 1, 3 \\ 0.5 & if \quad x = 5 \\ 0.7 & if \quad x = 4 \\ 0.3 & if \quad x = 7, 9 \end{cases}$$

2.2 Intuitionistic Fuzzy set:

The concept of fuzzy sets give the degree of membership of an element in a given set," intuitionistic fuzzy set (*IFS*)" gives both a degree of membership and a degree of non-membership. As for fuzzy sets, the degree of membership is a real number between 0 and 1. This also the case for the degree of non-membership, and further the sum of these two degrees is not greater than 1.

Definition 2.2.1:

An intuitionistic fuzzy set A, in S is an object of the following form $A = \{\langle x, \mu_A(x), v_A(x)\rangle | x \in S\}$ or simply $\langle \mu_A, v_A \rangle$. Where the function $\mu_A: S \to [0,1]$ and $v_A: S \to [0,1]$ define the degree of membership and the degree of non-membership, of the element, for each $x \in S$, $0 \le \mu_A(x) + v_A(x) \le 1$.

Remark 2.2.1:

.1. If $\pi_A(x) = 1 - \mu_A(x) - v_A(x)$, then $\pi_A(x)$ is called the degree of nondeterminance, of an element $x \in S$, to the set A

[Clearly $\pi_A(x) \in [0,1]\ \forall\, x \in S$]. .2.

Fuzzy set is particular case of the intuitionistic fuzzy set.

.3. If A is fuzzy set, then $\pi_A(x) = 0\ \ \forall\, x \in S$.

Operations on intuitionistic fuzzy set 2.2.1:

For every two " *IFS*" $A = \langle \mu_A, v_A \rangle$ and $B = \langle \mu_B, v_B \rangle$ we define the following relations:

.1. $A \subset B \Leftrightarrow \mu_A(x) \le \mu_B(x)$ and $v_A(x) \ge v_B(x), \forall\, x \in S$.

.2. $A = B \Leftrightarrow A \subset B\ \&\ B \subset A$.

.3. $\bar{A} = \langle v_A, \mu_A \rangle$

.4. $A \cap B = \langle \min(\mu_A, \mu_B), \max(v_A, v_B) \rangle = \langle \mu_A \wedge \mu_B, v_A \vee v_B \rangle$.

.5. $A \cup B = \langle \max(\mu_A, \mu_B), \min(v_A, v_B) \rangle = \langle \mu_A \vee \mu_B, v_A \wedge v_B \rangle$

.6. $A + B = \langle \mu_A + \mu_B - \mu_A . \mu_B, v_A . v_B \rangle$

.7. $A \,.\, B = \langle \mu_A . \mu_B, v_A + v_B - v_A . v_B \rangle$.

.8. $C(A) = \{\langle x, K, L\rangle | x \in S\}$ where

$K = max_{x \in S} \mu_A(x)$, $L = min_{x \in S} v_A(x)$.

.9. $I(A) = \{\langle x, K^*, L^* \rangle\} | x \in S\}$ where

$K^* = min_{x \in S} \mu_A(x)$, $L^* = max_{x \in S} v_A(x)$.

Definition 2.2.2:

The necessity operators is defined as :

$\square$ A $= \langle \mu_A, 1 - \mu_A \rangle$

$\Diamond$ A $= \langle 1 - v_A, v_A \rangle$.

Example 2.2.1:

Consider $S = \{a_1, a_2, a_3, a_4, a_5\}$. Let A and B be two intuitionistic fuzzy sets of S given by

$A = \{\langle a_1, 0.4, 0.6\rangle, \langle a_2, 0.3, 0.7\rangle, \langle a_3, 1, 0\rangle, \langle a_4, 0.8, 0.1\rangle, \langle a_5, 0.5, 0.4\rangle\}$.

And

$B = \{\langle a_1, 0.4, 0.4\rangle, \langle a_2, 0.5, 0.2\rangle, \langle a_3, 0.6, 0.2\rangle, \langle a_4, 0.1, 0.7\rangle , \langle a_5, 0, 1\rangle\}$.

We see that

$\bar{A} = \{\langle a_1, 0.6, 0.4\rangle, \langle a_2, 0.7, 0.3\rangle, \langle a_3, 0, 1\rangle, \langle a_4, 0.1, 0.8\rangle , \langle a_5, 0.4, 0.5\rangle\}$.

$\bar{B} = \{\langle a_1, 0.4, 0.4\rangle, \langle a_2, 0.2, 0.5\rangle, \langle a_3, 0.2, 0.6\rangle, \langle a_4, 0.7, 0.1\rangle , \langle a_5, 1, 0\rangle\}$.

$A \cap B = \{\langle a_1, 0.4, 0.6\rangle, \langle a_2, 0.3, 0.7\rangle, \langle a_3, 0.6, 0.2\rangle\}$

$A \cup B = \{\langle a_1, 0.4, 0.4\rangle, \langle a_2, 0.5, 0.2\rangle, \langle a_3, 1, 0\rangle,$

$\langle a_4, 0.8, 0.1\rangle, \langle a_5, 0.5, 0.4\rangle\}$. $A + B =$
$\{\langle a_1, 0.64, 0.24\rangle, \langle a_2, 0.65, 0.14\rangle, \langle a_3, 1, 0\rangle,$

$\langle a_4, 0.82, 0.07, \langle a_5, 0.5, 0.4\rangle\}$.

$A.B$=$\{\langle a_1, 0.16, 0.76\rangle, \langle a_2, 0.15, 0.76\rangle, \langle a_3, 0.6, 0.2\rangle, \langle a_4, 0.08, 0.73\rangle,$

$\langle a_5, 0, 1\rangle\}$.$\square A$ =$\{\langle a_1, 0.4, 0.6\rangle, \langle a_2, 0.3, 0.7\rangle, \langle a_3, 1, 0\rangle, \langle a_4, 0.8, 0.2\rangle,$

$\langle a_5, 0.5, 0.5\rangle\}$.

$\square B = \{\langle a_1, 0.4, 0.6\rangle, \langle a_2, 0.5, 0.5\rangle, \langle a_3, 0.6, 0.4\rangle, \langle a_4, 0.1, 0.9\rangle, \langle a_5, 0, 1\rangle\}$.

$\lozenge\, A = \{\langle a_1, 0.4, 0.6\rangle, \langle a_2, 0.3, 0.7\rangle, a_3, 1, 0\rangle, \langle a_4, 0.9, 0.1\rangle, \langle a_5, 0.6, 0.4\rangle\}$.

$\lozenge\, B = \{\langle a_1, 0.6, 0.4\rangle, \langle a_2, 0.8, 0.2\rangle, \langle a_3, 0.8, 0.2\rangle, \langle a_4, 0.3, 0.7\rangle, \langle a_5, 0, 1\rangle\}$.

$C(A) = \{\langle a_3, 1, 0\rangle\}$.

$C(B) = \{\langle a_3, 0.6, 0.2\rangle\}$.

$I(A) = \{\langle a_2, 0.3, 0.7\rangle\}$.

$I(B) = \{\langle a_5, 0, 1\rangle\}$.

Theorem 2.2.1:

For every *IFS A*

.1. $\square A = \overline{\lozenge\, \bar{A}}$

.2. $\lozenge A = \overline{\square \overline{A}}$

.3. $\square A \subset A \subset \lozenge A$

.4. $\square\square A = \square A$

.5. $\square\lozenge A = \lozenge A$

.6. $\lozenge\square A = \square A$

.7. $\lozenge\lozenge A = \lozenge A$

<u>Theorem 2.2.2</u>:

For every two *IFS's* A and B:

.1. $\square(A \cup B) = \square A \cup \square B$

.2. $\lozenge(A \cup B) = \lozenge A \cup \lozenge B$

<u>Theorem 2.2.3:</u>

For every two *IFS's* A and B

.1. $\square(A \cap B) = \square A \cap \square B$

.2. $\lozenge(A \cap B) = \lozenge A \cap \lozenge B$

.3. $\square(A + B) = \square A + \square B$

.4. $\lozenge(A + B) = \overline{\lozenge A . \lozenge B}$

.5. $\square(A . B) = \square A . \square B$

.6. $\lozenge(A . B) = \overline{\lozenge A + \lozenge B}$

<u>Proof:</u>

We will prove only 1 & 6.

.1. $A \cap B = \{\langle x, min\big(\mu_A(x), \mu_B(x)\big), max\big(v_A(x), v_B(x)\big)\rangle|\big\rangle x \in S\}.$

$= \{\langle x, min(\mu_A(x), \mu_B(x))\rangle | x \in S\}.$

$= \{\langle x, \mu_A(x)\rangle | x \in S\} \cap \{\langle x, \mu_B(x)\rangle | x \in S\}.$

$= \square A \cap \square B.$ ■

.6. $\lozenge\,(A.B) = \lozenge\,\{\langle x, \mu_A(x).\mu_B(x), v_A(x) + v_B(x) - v_A(x).v_B(x)\rangle |$

$x \in S\}.$

$= \{\langle x, 1 - v_A(x) - v_B(x) + v_A(x).v_B(x)\rangle | x \in S\}.$

$= \overline{\{\langle x, v_A(x) + v_B(x) - v_A(x).v_B(x)\rangle | x \in S\}}$

$= \overline{\{\langle x, v_A(x)\rangle | x \in S\} + \{\langle x, v_B(x)\rangle | x \in S\}}$

$= \overline{\lozenge\, A + \lozenge\, B}$ ■

Definition 2.2.3:

$A \subset_{\square} B$ Iff $(\,\forall\, x \in S)(\mu_A(x) \leq \mu_B(x))$,

$A \subset_{\lozenge} B$ Iff $(\,\forall\, x \in S)\big(v_A(x) \geq v_B(x)\big)$.

Theorem 2.2.4:

For every two *IFS's* A and B

.1. $A \subset_{\square} B$ iff $\square A \subset \square B$

.2. $A \subset_{\lozenge} B$ iff $\lozenge A \subset \lozenge B$

Proof :

We prove only 1.

.1. Let $A \subset_{\square} B, i.e. (\forall x \in S)\big(\mu_A(x) \le \mu_B(x)\big)$. Then for $\square A$ and $\square B$ it follows that $\square A \subset \square B$.

Conversly,

If $\square A \subset \square B$, then $(\forall x \in S)(\mu_A(x) \le \mu_B(x), i.e. A \subset_{\square} B)$. ■

Theorem 2.2.5:

For every two *IFS's* A and B

.1. $A \subset_{\square} B$ & $A \subset_{\lozenge} B$ iff $A \subset B$

.2. If $A \subset_{\square} B$ & $A \square B$, then $A \subset B$

.3. If $A \subset_{\lozenge} B$ & $B \square A$, then $A \subset B$

Proof :

.1. If $A \subset_{\square} B$ & $A \subset_{\lozenge} B$ then

$(\forall x \in S)(\mu_A(x) \le \mu_B(x))$ & $(\forall x \in S)(v_A(x) \ge v_B(x))$.

Hence $(\forall x \in S)(\mu_A(x) \le \mu_B(x)$ & $v_A(x) \ge v_B(x))$.

Therefore $A \subset B$.

Conversly, If $A \subset B$, then

$(\forall x \in S)\big(\mu_A(x) \le \mu_B(x)$ & $v_A(x) \ge v_B(x)\big)$,

$i.e. A \subset_{\square} B$ & $A \subset_{\lozenge} B$. ■

Theorem 2.2.6:

For every two $IFS's$ A and B

.1. $C(A)$ and $I(A)$ are $IFS's$

.2. $I(A) \subset A \subset C(A)$

.3. $C(A \cup B) = C(A) \cup C(B)$

.4. $C(C(A)) = C(A)$

.5. $C(\overline{0}) = \overline{0}$ where $\overline{0} = \langle 0, 1 \rangle$

Theorem 2.2.7:

For every IFS A

.1. $\lozenge A \subset C(A)$.

.2. $I(A) \subset \square A$.

.3. $\overline{I(\bar{A})} = C(A)$.

Definition 2.2.4:

Let $A = \{\langle x_1, \mu_A(x_1), v_A(x_1)\rangle | x_1 \in X_1\}$,

$B = \{\langle x_2, \mu_A(x_2), v_A(x_2)\rangle | \ x_2 \in X_2\}$ be two $IFS's$. The Cartesian product of these two $IFS's$ is the set

$$A \times B = \{\langle\langle x_1, x_2\rangle, \mu_A(x_1).\mu_B(x_2), v_A(x_1).v_B(x_2)\rangle | \ x_1 \in X_1, x_2 \in X_2\}$$

Remark 2.2.2:

Because:

$$0 \leq \mu_A(x_1).\mu_B(x_2) + v_A(x_1).v_B(x_2) \leq \mu_A(x_1) + v_A(x_1) \leq 1$$

it follows that $A \times B$ is an *IFS*.

Theorem 2.2.8:

For X_1, X_2, X_3 and for *IFS's* A, B(over X_1), C(over X_2), D(over X_3).

.1. $A \times C = C \times A$

.2. $(A \times C) \times D = A \times (C \times D)$

.3. $(A \cup B) \times C = (A \times C) \cup (B \times C)$

.4. $(A \cap B) \times C = (A \times C) \cap (B \times C$

.5. $(A + B) \times C \subset (A \times C) + (B \times C)$

.6. $(A \,.\, B) \times C \supset (A \times C) \,.\, (B \times C)$

Proof :

We prove only 5.

.5. $(A + B) \times C =$
$\{\langle x, \mu_A(x) + \mu_B(x) - \mu_A(x).\mu_B(x), v_A(x).v_B(x) | x \in X\} \times C$
$=\{\langle\langle x, y\rangle, \big(\mu_A(x) + \mu_B(x) - \mu_A(x).\mu_B(x)\big).\mu_C(y), v_A(x).v_B(x).v_C(y)\rangle \,| x \in X_1, y \in X_2\}$.

However
$\mu_A(x).\mu_C(y) + \mu_B(x).\mu_C(y) - \mu_A(x).\mu_B(x).\mu_C(y) \leq \mu_A(x).\mu_C(y) + \mu_B(x).\mu_C(y) - \mu_A(x).\mu_B(x).\mu_C(y)^2$
and

$v_A(x).v_B(x).v_C(y) \geq v_A(x).v_B(x).v_C(y)^2$, so that

$$(A+B)\times C \subset \{\langle\langle x,y\rangle, \mu_A(x).\mu_C(y)+\mu_B(x).\mu_C(x)-\mu_A(x).\mu_B(x).\mu_C(y)^2, v_A(x).v_B(x).v_C(y)^2\rangle | x\in X_1, y\in X_2\}$$

$$=\{\langle\langle x,y\rangle, \mu_A(x).\mu_C(y), v_A(x).v_C(y)\rangle | x\in X_1, y\in X_2\}$$

$$+\{\langle\langle x,y\rangle, \mu_B(x).\mu_C(y), v_B(x).v_C(y) | x\in X_1, y\in X_2\} \qquad =(A\times C)+(B\times C)$$

Theorem 2.2.9:

Let *A,B IFS's* in X_1,X_2, respectively then:

.1. $\square(A\times B) \subset \square A\times \square B$

.2. $\Diamond(A\times B) \supset \Diamond A\times \Diamond B$

.3. $C(A\times B) \subset C(A)\times C(B)$

.4. $I(A\times B) \supset I(A)\times I(B)$

Proof :

We prove only 1.

[1]. $\square(A\times B) =$

$$\square\{\langle\langle x,y\rangle, \mu_A(x), \mu_B(y), v_A(x).v_B(y)\rangle | x\in X_1, y\in X_2\}.$$

$$=\left\{\left\langle\langle x,y\rangle, \mu_A(x).\mu_B(y), 1-\mu_B(x).\mu_B(t)\right\rangle\middle| x\in X_1, y\in X_2\right\}.$$

$$\subset \{\langle x,y\rangle,\ \mu_A(x).\mu_B(x), \left(1-\mu_A(x)\right).\left(1-\mu_B(y)\right) | x\in\ X_1 , y\in X_2\}.$$

$$=\{\langle x,\mu_A(x), 1-\mu_A(x)\rangle | x\in X_1\}\times\{\langle y,\mu_B(y), 1-\mu_B(y)\rangle | y\in X_2\}.$$

$=\square A\times\square B.$

Theorem 2.2.10:

If A and B are two $IFS's$ in S, then

.1. $(A^C)^C = A$

.2. $(\bar{A})^C = \overline{(A^C)}$

.3. $(A \cup B)^C = A^C \cap B^C$

.4. $(A \cap B)^C = A^C \cup B^C$

Proof :

We prove only 4.

.4. $A^C \cup B^C = \{\langle x, \mu_A^C(x), v_A^C(x)\rangle | x \in S\} \cup \{\langle x, \mu_B^C(x), v_B^C(x)\rangle | x \in S\}.$

$= \{\langle x, \mu_A^C(x) \vee \mu_B^C(x), v_A^C(x) \wedge v_A^C(x) | x \in S\}.$

$= \{\langle x, \mu_{A\cap B}^C(x), v_{A\cup B}^C(x)\rangle | x \in S\}.$

$= \{\langle x, \mu_{(A\cap B)}(x), v_{(A\cup B)}(x)\rangle x \in S\}^C.$

$= (A \cap B)^C$ ■

Definition 2.2.5:

Let Φ be a mapping form a set X to a set Y. Let $A = \langle \mu_A, v_A\rangle$ be an intuitionistic fuzzy subset of X and $B = \langle \mu_B, v_B\rangle$ be an intuitionistic fuzzy subset of Y. Then, the inverse image $\Phi^{-1}(B) = (\Phi^{-1}(\mu_B), \Phi^{-1}(v_B))$ is the intuitionistic fuzzy subset of X defined by

$$\Phi^{-1}(B)(x) = \left(\Phi^{-1}(\mu_B)(x), \Phi^{-1}(v_B)(x)\right) = \left(\mu_B\left(\Phi(x)\right), v_B\left(\Phi(x)\right)\right).$$

The image $\Phi(A) = (\Phi(\mu_A), \Phi(v_A))$ is the intuitionistic fuzzy subset of Y defined by:

$$\Phi(\mu_A)(x) = \begin{cases} \sup\{\mu_A(t) : t \in \Phi^{-1}(y)\} & if \quad \Phi^{-1}(y) \neq \emptyset \\ 0 & otherwise \end{cases}$$

And

$$\Phi(v_A)(x) = \begin{cases} \inf\{v_A(t) : \ t \in \Phi^{-1}(y)\} & if \quad \Phi^{-1}(y) \neq \emptyset \\ 1 & otherwise \end{cases}$$

Chapter Three: Intuitionistic Fuzzy Interior Ideals Of Γ-Semigroups

3.1 Intuitionistic fuzzy subsemigroups :

In this chapter we study the intuitionistic fuzzication on the concept of several ideals in a semigroup S, and investigate some properties of such ideals.

The materials of this chapter is taken from the following references:[12], [13], [14].

Definition 3.1.1 :

An IFS $A = \langle \mu_A, v_A \rangle$ in S is called an intuitionistic fuzzy subsemigroup of S if:

(i) $\mu_A(xy) \geq inf\{\mu_A(x), \mu_A(y)\}$.

(ii) $v_A(xy) \leq sup\{v_A(x), v_A(y)\}$.

Example 3.1.1:

Consider the semigroup S=($\mathbb{Z}$, $+$), let $\boldsymbol{E}$ be the set of all even integers, and $\boldsymbol{O}$ be the set of all odd integers. Define $\mu_A: S \rightarrow [0,1]$ by:

$$\mu_A(x) = \begin{cases} 0.6 & if \quad x \in E \\ 0.4 & if \quad x \in O, \end{cases}$$

And define $v_A: S \rightarrow [0,1]$ by:

$$v_A(x) = \begin{cases} 0.5 & if \quad x \in E \\ 0.7 & if \quad x \in O. \end{cases}$$

(i) If $x, y \in E$, then $\mu_A(x) = \mu_A(y) = 0.6$ and $\mu_A(x + y) = 0.6$,$v_A(x) = v_A(y) = 0.5$ and $v_A(x + y) = 0.5$.

(ii) If $x, y \in O$, then $\mu_A(x) = \mu_A(y) = 0.4$ and $\mu_A(x+y) = 0.6$, $v_A(x) = v_A(y) = 0.7$ and $v_A(x+y) = 0.5$.

(iii) If $x \in E, y \in O$, then $\mu_A(x) = 0.6, \mu_A(y) = 0.4$ and $\mu_A(x+y) = 0.4, v_A(x) = 0.5, v_A(y) = 0.7,\ v_A(x+y) = 0.7$.

Hence $\mu_A(x+y) \geq \inf\{\mu_A(x), \mu_A(y)\}$, $v_A(x+y) \leq \sup\{v_A(x), v_A(y)\}$, for all $x, y \in S$.

Thus $A = \langle \mu_A, v_A \rangle$ is an intuitionistic fuzzy subsemigroup of the semigroup $\mathbb{Z}$.

Example 3.1.2:

Let $S = \{0, 1, 2, 3, 4\}$ be a semigroup with the following Cayley table:

Table 3.1.1

.	0	1	2	3	4
0	0	0	0	0	0
1	0	1	0	3	0
2	0	0	2	0	4
3	0	3	0	0	1
4	0	0	4	2	0

Define an IFS $A = \langle \mu_A, v_A \rangle$ in S by $\mu_A(0) = \mu_A(1) = \mu_A(2) = 0.6,\ \mu_A(3) = \mu_A(4) = 0.2$, $v_A(0) = v_A(1) = v_A(2) = 0.2$ and $v_A(3) = v_A(4) = 0.6$.

Then $A = \langle \mu_A, v_A \rangle$ is an intuitionistic fuzzy subsemigroup of S.

Definition 3.1.2:

An IFS $A = \langle \mu_A, v_A \rangle$ in S is called an:

(i) Intuitionistic fuzzy left ideal of S if:

$\mu_A(xy) \geq \mu_A(y)$ and $v_A(xy) \leq v_A(y)$ for any $x, y \in S$.

(ii) Intuitionistic fuzzy right ideal of S if:

$\mu_A(xy) \geq \mu_A(x)$ and $v_A(xy) \leq v_A(x)$ for any $x, y \in S$.

(iii) Intuitionistic fuzzy (two-sided) ideal of S if it is both an intuitionistic fuzzy left and an intuitionistic fuzzy right ideal of S.

Example 3.1.3:

Consider the semigroup $S = \mathbb{R}$.

Define $\mu: S \to [0,1]$, $v: S \to [0,1]$ by:

$$\mu_A(x) = \begin{cases} 0.5 & if \quad x \in \mathbb{Q} \\ 0.3 & if \quad x \notin \mathbb{Q}, \end{cases}$$

$$v_A(x) = \begin{cases} 0.2 & if \quad x \in \mathbb{Q} \\ 0.7 & if \quad x \notin \mathbb{Q}. \end{cases}$$

Then $A = \langle \mu_A, v_A \rangle$ is an intuitionistic fuzzy left ideal.

Example 3.1.4:

Let $S = \{1, 2, 3\}$ be the semigroup with the following Cayley table:

$\circ$	1	2	3
1	3	3	3
2	3	3	3
3	3	3	3

Table 3.1.2

Define an IFS $A = \langle \mu_A, v_A \rangle$ in S by:

$\mu_A(1) = \mu_A(2) = \mu_A(3) = 0.5$,

$v_A(1) = v_A(2) = v_A(3) = 0.3$.

It is clear that $A = \langle \mu_A, v_A \rangle$ is an intuitionistic fuzzy ideal of S.

Remark 3.1.1:

It is clear that any intuitionistic fuzzy left (right) ideal of S is an intuitionistic fuzzy subsemigroup of S.

Example 3.1.5:

This example shows that an intuitionistic fuzzy subsemigroup need not be an intuitionistic fuzzy right ideal.

Consider a semigroup $S = \{1, 2\}$ with the following Cayley table:

.	1	2
1	1	2
2	2	2

Table 3.1.3

Define an IFS $A = \langle \mu_A, v_A \rangle$ in S by:

$\mu_A(1) = 0.6$, $\mu_A(2) = 0.4$,

$v_A(1) = 0.2$, $v_A(2) = 0.5$.

So $A = \langle \mu_A, v_A \rangle$ is an intuitionistic fuzzy subsemigroup of S but $A = \langle \mu_A, v_A \rangle$ is not an intuitionistic fuzzy right ideal of S.

Definition 3.1.3:

An intuitionistic fuzzy subsemigroup $A = \langle \mu_A, v_A \rangle$ of S is called an intuitionistic fuzzy bi-ideal of S if:

(i) $\mu_A(xwy) \geq \inf\{\mu_A(x), \mu_A(y)\}$

(ii) $v_A(xwy) \leq \sup\{\mu_A(x), \mu_A(y)\}$

For all $w, x, y \in S$.

Example 3.1.6:

Let S ={1, 2, 3, 4, 5} be a semigroup with the following Cayley table:

.	1	2	3	4	5
1	1	1	1	1	1
2	1	1	1	1	1
3	1	1	3	3	5
4	1	1	3	4	5
5	1	1	3	3	5

Table 3.1.4

Define IFS $A = \langle \mu_A, v_A \rangle$ in S by: $\mu_A(1) = 0.6, \mu_A(2) = 0.5, \mu_A(3) = 0.4, \mu_A(4) = 0.4, \mu_A(5) = 0.3,$ $v_A(1) = v_A(2) = 0.3, v_A(3) = 0.4, v_A(4) = 0.5, v_A(5) = 0.6$.

By routine calculation, it is clear that $A = \langle \mu_A, v_A \rangle$ is an intuitionistic fuzzy bi-ideal of S.

Theorem 3.1.1:

Every intuitionistic fuzzy left (right)ideal of S is an intuitionistic fuzzy bi-ideal of S.

Proof:

Let $A = \langle \mu_A, v_A \rangle$ be an intuitionistic fuzzy left ideal of S and $w, x, y \in S$. Then

$$\mu_A(xwy) = \mu_A(xw)y) \geq \mu_A(y) \geq \inf\{\mu_A(x), \mu_A(y)\},$$

And

$$v_A(xwy) = v_A((xw)y) \leq v_A(y) \leq \sup\{v_A(x), v_A(y)\}.$$

Thus $A = \langle \mu_A, v_A \rangle$ is an intuitionistic fuzzy bi-ideal of S . The right case is proved in an analongous way. ■

Theorem 3.1.2:

Let S be a regular semigroup. If every bi-ideal of S is a right (left) ideal of S,then every intuitionistic fuzzy bi-ideal of S is an intuitionistic fuzzy right (left) ideal of S.

Theorem 3.1.3:

If $\{A_i\}_{i\in\wedge}$ is a family of intuitionistic fuzzy bi-ideal of S, then $\cap A_i$ is an intuitionistic fuzzy bi-ideal of S, where $\cap A_i = (\wedge \mu_{A_i}, \vee v_{A_i})$ and

$$\wedge \mu_{A_i}(x) = \inf\{\mu_{A_i}(x) | i \in \wedge,\ x \in S\},$$
$$\vee v_{A_i}(x) = \sup\{v_{A_i}(x) | i \in \wedge,\ x \in S\}.$$

Theorem 3.1.4:

If an IFS $A = \langle \mu_A, v_A \rangle$ in S is an intuitionistic fuzzy bi-ideal of S, then so is $\Box A = \langle \mu_A, \mu_A^C \rangle$.

Definition 3.1.4:

An intuitionistic fuzzy subsemigroup $A = \langle \mu_A, \nu_A \rangle$ of S is called an intuitionistic fuzzy (1, 2)-ideal of S if:

(i) $\mu_A(xw(yz)) \geq \inf\{\mu_A(x), \mu_A(y), \mu_A(z)\}$,

(ii) $\nu_A\big(xw(yz)\big) \leq \sup\{\nu_A(x), \nu_A(y), \nu_A(z)\}$, For all $w, z, x, y \in S$.

Example 3.1.7:

Let S ={1, 2, 3, 4, 5} be a semigroup with the following Cayle table:

.	1	2	3	4	5
1	1	1	1	1	1
2	1	1	1	2	3
3	1	2	3	1	1
4	1	1	1	4	5
5	1	4	5	1	1

Table 3.1.5

Define an IFS $A = \langle \mu_A, \nu_A \rangle$ in S by:

$\mu_A(1) = \mu_A(2) = \mu_A(3) = 1, \mu_A(4) = \mu_A(5) = 0$, and $\nu_A(1) = \nu_A(2) = \nu_A(3) = 0, \nu_A(4) = \nu_A(5) = 1$.

By routine calculation, it is clear that $A = \langle \mu_A, \nu_A \rangle$ is an intuitionistic fuzzy (1, 2)-ideal of S.

We note that $M = \{1, 2, 3\}$ is a $(1, 2)$-ideal of S , hence $A = \langle \mu_A, v_A \rangle$ can be defined as follows:

$$\mu_A(x) = \begin{cases} 1 & if \quad x \in M \\ 0 & other\ wise \end{cases} \quad \text{and}$$
$$v_A(x) = \begin{cases} 0 & if \quad x \in M \\ 1 & other\ wise. \end{cases}$$

Theorem 3.1.5:

Every intuitionistic fuzzy bi-ideal is an intuitionistic fuzzy (1, 2)-ideal.

Theorem 3.1.6:

If S is a regular semigroup, then every intuitionistic fuzzy $(1, 2)$-ideal of S is an intuitionistic fuzzy bi-ideal of S.

Theorem 3.1.7:

Let $A = \langle \mu_A, v_A \rangle$ be an intuitionistic fuzzy bi-ideal of S. If S is a completely regular, then $A(a) = A(a^2)$ for all $a \in S$.

Theorem 3.1.8:

Let $A = \langle \mu_A, v_A \rangle$ be an intuitionistic fuzzy ideal of S.If S is an intra-regular, then $A(a) = A(a^2)$ for all $a \in S$.

Theorem 3.1.9:

Let $A = \langle \mu_A, v_A \rangle$ be an intuitionistic fuzzy ideal of S. If S is an intra-regular, then $A(ab) = A(ba)$ for all $a, b \in S$.

Theorem 3.1.10:

An IFS $A = \langle \mu_A, \nu_A \rangle$ is an intuitionistic fuzzy bi-ideal of S if and only if the fuzzy sets μ_A and ν_A^C are fuzzy bi-ideals of S.

Corollary 3.1.1:

An IF $A = \langle \mu_A, \nu_{\mathrm{A}} \rangle$ is an intuitionistic fuzzy bi-ideal of S if and only if $\square A = \langle \mu_A, \mu_A^C \rangle$ and $\lozenge A = \langle \nu_A, \nu_A^C \rangle$ are intuitionistic fuzzy bi-ideal of S.

Theorem 3.1.11:

Let $f: S \to T$ be a homomorphism of semigroups. If $B = \langle \mu_B, \nu_B \rangle$ is an intuitionistic fuzzy bi-ideal of T, then the preimage $f^{-1}(B) = (f^{-1}(\mu_{\mathrm{B}}), f^{-1}(\nu_{\mathrm{B}}))$ of B under f is an intuitionistic fuzzy bi-ideal of S.

3.2 Intuitionistic fuzzy interior ideals of Γ-semigroup:

In this section we study the notion of intuitionistic fuzzy interior ideals and intuitionistic fuzzy characteristic interior ideals of Γ-semigroup and obtain some of their properties such as the characteristic function criterion and level subset criterion. Finally, in this section we will see characterization of simple Γ-semigroup in terms of intuitionistic fuzzy interior ideal.

The materials of this chapter is taken from the following reference:[1].

Definition 3.2.1:

A non-empty intuitionistic fuzzy subset $A = \langle \mu_A, \nu_A \rangle$ of Γ-semigroup S is called an intuitionistic fuzzy subsemigroup of S if it satisfies:

(i) $\mu_A(x\gamma y) \geq \min\{\mu_A(x), \mu_A(y)\}, \forall\, x, y \in S \text{ and } \forall \gamma \in \Gamma,$

(ii) $\nu_A(x\gamma y) \leq \max\{\nu_A(x), \nu_A(y)\}, \forall x, y \in S \text{ and } \forall \gamma \in \Gamma.$

Dentition 3.2.2:

A Γ-semigroup is said to be intuitionistic fuzzy simple if every intuitionistic fuzzy ideal of S is a constant function.

Lemma 3.2.1:

A Γ-semigroup S is simple if and only if it is intuitionistic fuzzy simple.

Proof:

Let S be a Γ-semigroup and $A = \langle \mu_A, v_A \rangle$ be an intuitionistic fuzzy interior ideal of S. Let $a, b \in S$. Then, by theorem 3.8[21], $U\big(\mu_A, \mu_A(a)\big), U\big(\mu_A, \mu_A(b)\big), L\big(v_A, v_A(a)\big)$ and $L(v_A, v_A(b))$ are ideals of S. Since S is simple, we have $U\big(\mu_A, \mu_A(a)\big) = S = U(\mu_A, \mu_A(b))$ and $L\big(v_A, v_A(a)\big) = S = L(v_A, v_A(b))$. Consequently, it follows that $a \in U\big(\mu_A, \mu_A(b)\big), L(v_A, v_A(b))$ and $b \in U\big(\mu_A, \mu_A(a)\big), L(v_A, v_A(a))$, which implies that $\mu_A(a) \geq \mu_A(b), v_A(a) \leq v_A(b)$ and $\mu_A(b) \geq \mu_A(a), v_A(b) \leq v_A(a)$. So, $\mu_A(a) = \mu_A(b)$ and $v_A(a) = v_A(b)$ for all $a, b \in S$. Hence, S is intuitionistic fuzzy simple.

Conversely, suppose that S is intuitionistic fuzzy simple. Let I be an ideal of S. Then, $(\varkappa_I, \varkappa_I^c)$ is an intuitionistic fuzzy ideal of S(cf. Corollary 3.11[21]). Let $x \in S$.

Since S is intuitionistic fuzzy simple, the intuitionistic fuzzy ideal $(\varkappa_I, \varkappa_I^c)$ is a constant function, and so $\mu_I(x) = \mu_I(a)$ and $v_I(x) = v_I(a)$ $\forall a \in I$. Hence, $\big(\varkappa_I(x), \varkappa_I^c(x)\big) = (1, 0)$, whence $x \in I$. Thus, we have $S = I$. Hence, the Γ-semigroup is simple. ■

Definition 3.2.3:

An intuitionistic fuzzy subsemigroup $A = \langle \mu_A, \nu_A \rangle$ of Γ-semigroup S is called an intuitionistic fuzzy interior ideal of S if it satisfies: (i) $\mu_A(x\alpha a\beta y) \geq \mu_A(a), \ \forall\, x, y \in S \ \ and \ \ \forall\, \gamma \in \Gamma$,

(ii) $\nu_A(x\alpha a\beta y) \leq \nu_A(a), \ \forall\, x, y \in S \ \ and \ \ \forall\, \gamma \in \Gamma$.

Definition 3.2.4:

(1) A non-empty intuitionistic fuzzy subset $A = \langle \mu_A, \nu_A \rangle$ of Γ-semigroup S is called an intuitionistic fuzzy left ideal of S if it satisfies:

(i) $\mu_A(x\gamma y) \geq \mu_A(y), \ \forall\, x, y \in S \ \ and \ \ \forall\, \gamma \in \Gamma$,

(ii) $\nu_A(x\gamma y) \leq \nu_A(y), \ \forall\, x, y \in S \ \ and \ \ \forall\, \gamma \in \Gamma$.

(2) A non-empty intuitionistic fuzzy subset $A = \langle \mu_A, \nu_A \rangle$ of Γ-semigroup S is called an intuitionistic fuzzy right ideal of S if it satisfies:

(i) $\mu_A(x\gamma y) \geq \mu_A(x), \ \forall\, x, y \in S \ \ and \ \ \forall\, \gamma \in \Gamma$,

(ii) $\nu_A(x\gamma y) \leq \nu_A(x), \ \forall\, x, y \in S \ \ and \ \ \forall\, \gamma \in \Gamma$.

(3) A non-empty intuitionistic fuzzy subset $A = \langle \mu_A, \nu_A \rangle$ of Γ-semigroup S is called an intuitionistic fuzzy ideal of S if it is an intuitionistic fuzzy left and intuitionistic fuzzy right ideal of S.

Example 3.2.1:

Let S be the set of all non-positive integers and Γ be the set of all non-positive even integers. Then , S is a Γ-semigroup where $a\gamma b$ denoted the usual multiplication of integers a, γ, b with $a, b \in S$ and $\gamma \in \Gamma$. Let $A = \langle \mu_A, \nu_A \rangle$ be an intuitionistic fuzzy subset of S, defined as follows:

$$\mu_A = \begin{cases} 1 & if \quad x = 0 \\ 0.1 & if \quad x = -1, -2 \end{cases}, \mu_A = \begin{cases} 0 & if \quad -3, -6 \\ 0.4 & otherwise \end{cases}$$

And

$$\nu_A = \begin{cases} 0 & if \quad x = 0 \\ 0.8 & if \quad x = -1, -2 \end{cases}, \nu_A = \begin{cases} 1 & if \quad x = -3, -6 \\ 0.5 & otherwise \end{cases}$$

Then, the intuitionistic fuzzy subset $A = \langle \mu_A, \nu_A \rangle$ of S is an intuitionistic fuzzy interior ideal of S which is not an intuitionistic fuzzy ideal of S.

Proposition 3.2.1:

If $\{A_i\}_{i \in \Lambda}$ is a family of intuitionistic fuzzy interior ideals of a Γ-semigroup S, then $\bigcap_{i \in \Lambda} A_i$ is an intuitionistic fuzzy interior ideal of S, provided it is non-empty.

Proof :

Let $B = \bigcap_{i \in \Lambda} A_i$ and $x, a, y \in S, \alpha, \beta, \gamma \in \Gamma$. Then,

$$\mu_B(x\gamma y) = inf_{i \in \Lambda}\{\mu_{A_i}(x\gamma y)$$

$$\geq inf_{i \in \Lambda}\{\min\{\mu_{A_i}(x), \mu_{A_i}(y)\}\}$$

$$= \min\{ inf_{i \in \Lambda}\{\mu_{A_i}(x), inf_{i \in \Lambda}\{\mu_{A_i},(y)\} \}$$

$$= \min\{\mu_B(x), \mu_B(y)\}$$

Also, we have

$$\nu_B(x\gamma y) = sup_{i \in \Lambda}\{\nu_{A_i}(x\gamma y)\}$$

$$\leq sup_{i \in \Lambda}\{\max\{\nu_{A_i}(x), \nu_{A_i}(y)\}\}$$

$$= \max\{sup_{i \in \Lambda}\{\nu_{A_i}(x)\}, sup_{i \in \Lambda}\{\nu_{A_i}(y)\} \}$$

$$= \max\{\nu_B(x), \nu_B(y)\}.$$

Hence, $\bigcap_{i \in \Lambda} A_i$ is an intuitionistic fuzzy subsemigroup of S. Again,

$$\mu_B(a) = inf_{i\in\wedge}\{\mu_{A_i}(a)\} \le inf_{i\in\wedge}\{\mu_{A_i}(x\alpha a\beta y) = \mu_B(x\alpha a\beta y).$$

Also, we have

$$v_B(a) = sup_{i\in\wedge}\{v_{A_i}(a) \ge sup_{i\in\wedge}\{v_{A_i}(x\alpha a\beta y) = v_B(x\alpha a\beta y).$$

Therefore, $\bigcap_{i\in\wedge} A_i$ is an intuitionistic fuzzy interior ideal of S. ■

Lemma 3.2.2:

If $A = \langle \mu_A, v_A \rangle$ is an intuitionistic fuzzy interior ideal of S, then so is $\square A = \langle \mu_A, \mu_A^C \rangle$.

Lemma 3.2.3:

If $A = \langle \mu_A, v_A \rangle$ is an intuitionistic fuzzy interior ideal of S, then so is $\lozenge A = \langle v_A^C, v_A \rangle$.

Theorem 3.2.1:

$A = \langle \mu_A, v_A \rangle$ is an intuitionistic fuzzy interior ideal of S if and only if $\square A$ and $\lozenge A$ are intuitionistic fuzzy interior ideals of S.

Corollary 3.2.1:

An intuitionistic fuzzy subset $A = \langle \mu_A, v_A \rangle$ of S is an intuitionistic fuzzy interior ideal of S if and only if μ_A and v_A^C are fuzzy interior ideals of S.

Definition 3.2.5:

For any $t \in [0,1]$ and a fuzzy subset μ of S, the set $U(\mu;t) = \{x \in S : \mu(x) \geq t\}$(respectively, $L(\mu;t) = \{x \in S: \mu(x) \leq t\}$) is called an upper (respectively, lower) t-level cut of μ.

Example 3.2.2:

Let $S = \{a,b,c\}$ and $\Gamma = \{\gamma,\delta\}$, where γ, δ is defined on S with the following Caley table:

δ	a	b	c
a	a	a	a
b	b	b	b
c	c	c	c

and

Table 3.2.1 Table 3.2.2

Then, S is a Γ-semigroup. Let us consider a subset $\mu : S \rightarrow [0, 1]$, by $\mu(a)=0.8$, $\mu(b)= 0.7$, $\mu(c) =0.6$. For $t = 0.7$, $U(\mu;t) = \{a,b\}$ and $L(\mu;t) = \{b,c\}$.

Theorem 3.2.2:

If $A = \langle \mu_A, v_A \rangle$ is an intuitionistic fuzzy interior ideal of S, then the upper and lower level cuts $U(\mu_A;t)$ and $L(v_A;t)$ are interior ideals of S for every $t \in Im(\mu_A) \cap Im(v_A)$.

γ	a	b	c
a	a	a	a
b	b	b	b
c	c	c	c

Proof:

Let $t \in Im(\mu_A) \cap Im(v_A)$ and let $x,y \in U(\mu_A;t)$ and $\gamma \in \Gamma$. Then, $\mu_A(x) \geq t$ and $\mu_A(y) \geq t$. Now, since $A = \langle \mu_A, v_A \rangle$ is an intuitionistic fuzzy interior ideal of S, it is an intuitionistic fuzzy subsemigroup of S, hence

$\mu_A(x\gamma y) \geq \min\{\mu_A(x), \mu_A(x), \mu_A(y)\} \geq t$. Consequently, $x\gamma y \in U(\mu_A; t)$. Hence, $U(\mu_A; t)$ is a subsemigroup of S.

Again, let $x, y \in S, a \in U(\mu_A; t)$ and $\alpha, \beta \in \Gamma$. Then, $\mu_A(a) \geq t$.

Now, since $A = \langle \mu_A, v_A \rangle$ is an intuitionistic fuzzy interior ideal of S, hence $\mu_A(x\alpha a\beta y) \geq \mu_A(a) \geq t$. Consequently, $x\alpha a\beta y \in U(\mu_A; t)$. Hence, $U(\mu_A; t)$ is an interior ideal of S.

Similarly, let $x, y \in L(v_A; t)$ and $\gamma \in \Gamma$. Then, $v_A(x) \leq t$ and $v_A(y) \leq t$. Now, since $A = \langle \mu_A, v_A \rangle$ is an intuitionistic fuzzy interior ideal of S, it is an intuitionistic fuzzy subsemigroup of S, hence $v_A(x\gamma y) \leq \max\{v_A(x), v_A(y)\} \leq t$. Consequently, $x\gamma y \in L(v_A; t)$. Hence, $L(v_A; t)$ is a subsemigroup of S.

Again, let $x, y \in S$, $a \in L(v_A; t)$ and $\alpha, \beta \in \Gamma$. Then, $v_A(a) \leq t$

Now, since $A = \langle \mu_A, v_A \rangle$ is an intuitionistic fuzzy interior ideal of S, hence $v_A(x\alpha a\beta y) \leq v_A(a) \leq t$. Consequently, $x\alpha a\beta y \in L(v_A; t)$. Hence, $L(v_A; t)$ is an interior ideal of S. ■

Theorem 3.2.3

If $A = \langle \mu_A, v_A \rangle$ is an intuitionistic fuzzy subset of S such that the non-empty sets $U(\mu_A; t)$ and $L(v_A; t)$ are interior ideals of S for all $t \in [0, 1]$. Then, $A = \langle \mu_A, v_A \rangle$ is an intuitionistic fuzzy interior ideal of S.

Proof :

For $t \in [0, 1]$, let us assume that the non-empty sets $U(\mu_A; t)$ and $L(v_A; t)$ are interior ideals of S . Now, we shall that $A = \langle \mu_A, v_A \rangle$ satisfies the conditions of Definitions 2.1 and 2.2 . Let $x, y \in S$ and $\gamma \in \Gamma$. Let $t_0 =$

$\min\{\mu_A(x), \mu_A(y)\}$ and $t_1 = \max\{v_A(x), v_A(y)\}$. Then, $x, y \in U(\mu_A; t_0)$ and $x, y \in L(v_A; t_1)$. So $x\gamma y \in U(\mu_A; t_0)$ and $x\gamma y \in L(v_A; t_1)$ which implies that $\mu_A(x\gamma y) \geq t_0 = \min\{\mu_A(x), \mu_A(y)\}$ and $x\gamma y \leq t_1 = \max\{v_A(x), v_A(y)\}$.

Hence, $A = \langle \mu_A, v_A \rangle$ is an intuitionistic fuzzy subsemigruop of S .

Again, let $x, a, y \in S$ and $\alpha, \beta \in \Gamma$. Let $t_2 = \mu_A(a)$ and $t_3 = v_A(a)$. Then, $a \in U(\mu_A; t_2)$ and $a \in L(v_A; t_3)$. So $x\alpha a\beta y \in U(\mu_A; t_2)$ and $x\alpha a\beta y \in L(v_A; t_3)$ which implies that $\mu_A(x\alpha a\beta y) \geq t_2 = \mu_A(a)$ and $x\alpha a\beta y \leq t_3 = v_A(a)$. Hence, $A = \langle \mu_A, v_A \rangle$ is an intuitionistic fuzzy interior ideal of S .

■

Corollary 3.2.2:

Let I be an interior ideal of a Γ-semigroup S. If two fuzzy subsets μ and v are defined on S by

$$\mu(x) := \begin{cases} \alpha_0 & if \quad x \in I \\ \alpha_1 & if \quad x \in S - I \end{cases}$$

And

$$v(x) := \begin{cases} \beta_0 & if \quad x \in I \\ \beta_1 & if \quad x \in S - I \end{cases},$$

Where

$0 \leq \alpha_1 < \alpha_0$, $\leq \beta_0 < \beta_1$ and $\alpha_i + \beta_i \leq 1$ for $i = 0, 1$. Then, $A = \langle \mu, v \rangle$ is an intuitionistic fuzzy interior ideal of S and $U(\mu; \alpha_0) = I = U(v; \beta_0)$.

Corollary 3.2.3

Let χ_I be the characteristic function of an interior ideal I of S. Then, $I = (\chi_I, \chi_I^c)$ is an intuitionistic fuzzy interior ideal of S.

Example 3.2.3:

Let S denote the set of all integers of the form $4n+1$ and Γ dente the set of all integers of the form $4n+3$ where n is an integer. Then, S is a regular Γ-semigroup where $a+\alpha+b$ denote the usual sum of integers a, α, b with $a, b \in S$ and $\alpha \in \Gamma$. Let $A = \langle \mu_A, v_A \rangle$ be an intuitionistic fuzzy subset of S, defined as follows:

$$\mu_A(x) = \begin{cases} 1 & if \;\; x = 4n+1 \\ 0.3 & if \;\; x = 4n+3 \end{cases},$$

And

$$v_A(x) = \begin{cases} 0 & if \;\; x = 4n+1 \\ 0.6 & if \;\; x = 4n+3 \end{cases}$$

Then, $A = \langle \mu_A, v_A \rangle$ is an intuitionistic fuzzy interior ideal of S which is also an intuitionistic fuzzy ideal of S.

Proposition 3.2.2:

Let $\Phi\colon S \to T$ be a homomorphism of Γ-semigroups. Then,

(i) If $B = \langle \mu_B, v_B \rangle$ is an intuitionistic fuzzy interior ideal of T, then $\Phi^{-1}(B) = (\Phi^{-1}(\mu_B), \Phi^{-1}(v_B))$ is an intuitionistic fuzzy interior ideal of S.

(ii) If Φ is onto and $A = \langle \mu_A, v_A \rangle$ is an intuitionistic fuzzy interior ideal of S, then $\Phi(A) = (\Phi(\mu_A), \Phi(v_A))$ is an intuitionistic fuzzy interior ideal of T.

Proof :

(i)Let $B = \langle \mu_B, v_B \rangle$ be an intuitionistic fuzzy interior ideal of T. Then, $B = \langle \mu_B, v_B \rangle$ is an intuitionistic fuzzy subsemigroup of T. Let $x, y \in S$ and $\gamma \in \Gamma$. Then, $\Phi^{-1}(\mu_B)(x\gamma y) = \mu_B\big(\Phi(x\gamma y)\big) = \mu_B(\Phi(x)\gamma\Phi(y))$

(since Φ is a homomorphism o Γ-emigroups)

$\geq \min\{\mu_B\big(\Phi(x)\big), \mu_B\big(\Phi(y)\big)\}$

$= \min\{\Phi^{-1}(\mu_B)(x), \Phi^{-1}(\mu_B)(y)\}$

And $\Phi^{-1}(v_B)(x\gamma y) = v_B\big(\Phi(x\gamma y)\big) = v_B(\Phi(x)\gamma\Phi(y))$.

(since Φ is a homomorphism of Γ-semigroups)

$\leq \max\{v_B\big(\Phi(x)\big), v_B\big(\Phi(y)\big)\}$

$= \max\{\Phi^{-1}(v_B(x), \Phi^{-1}(v_B)(y)\}$.

Thus, $\Phi^{-1}(B) = (\Phi^{-1}(\mu_B), \Phi^{-1}(v_B))$ is an intuitionistic fuzzy subsemigroup of S. Again, let $x, y, a \in S$ and $\alpha, \beta \in \Gamma$. Then, $\Phi^{-1}(\mu_B)(x\alpha a\beta y) = \mu_B(\Phi(x\alpha a\beta y))$

$= \mu_B(\Phi(x)\alpha\Phi(a)\beta\Phi(y))$

(since Φ is a homomorphism of Γ-semigroups)

$$\geq \mu_B\big(\Phi(a)\big) = \Phi^{-1}(\mu_B)(a)$$

And

$$\Phi^{-1}(v_B)(x\alpha a\beta y) = v_B(\Phi(x\alpha a\beta y))$$

$$= v_B(\Phi(x)\alpha\Phi(a)\beta\Phi(y))$$

(since Φ is a homomorphism of Γ-semigroups)

$$\leq v_B\big(\Phi(a)\big) = \Phi^{-1}(v_B(a).$$

Therefore, $\Phi^{-1}(B) = (\Phi^{-1}(\mu_B), \Phi^{-1}(v_B))$ is an intuitionistic fuzzy interior ideal of S.

(ii) Let $A = \langle \mu_A, v_A \rangle$ be an intuitionistic fuzzy interior ideal of S. Then, it is an intuitionistic fuzzy subsemigroup of S. Since $(\Phi(\mu_A)(x') = sup_{\Phi(x)=x'}\{\mu_A(x)\}$

For $x' \in T$ and $\big(\Phi(v_A)\big)(x') = inf_{\Phi(x)=x'}\{v_A(x)\}$ for $x' \in T$, so $\Phi(A) = (\Phi(\mu_A), \Phi(v_A))$ is non-empty. Let $x', y' \in T$ and $\gamma \in \Gamma$. Then,

$$\big(\Phi(\mu_A)\big)(x'\gamma y') = sup_{\Phi(z)=x'\gamma y'}\{\mu_A(z)\}$$

$$\geq sup_{\Phi(x)=x',\Phi(y)=y'}\{\mu_A(x\gamma y)\}$$

$$= \min\{ sup_{\Phi(x)=x'}\{\mu_A(x)\}, sup_{\Phi(y)=y'}\{\mu_A(y)\}\}$$

$$= \min\{\big(\Phi(\mu_A)\big)(x'), \big(\Phi(\mu_A)\big)(y')\}$$

And

$$\big(\Phi(v_A)\big)(x'\gamma y') = inf_{\Phi(z)=x'\gamma y'}\{v_A(z)\}$$

$$\leq inf_{\Phi(x)=x',\Phi(y)=y'}\{v_A(x\gamma y)\}$$

$$\leq inf_{\Phi(x)=x',\Phi(y)=y'}\{\max\{v_A(x), v_A(y)\}\}$$

$$= \max\{inf_{\Phi(x)=x'}\{v_A(x)\}, inf_{\Phi(y)=y'}\{v_A(y)\}\}$$

$$= \max\{\big(\Phi(v_A)\big)(x'), \big(\Phi(v_A)\big)(y')\}$$

Therefore, $\Phi(A) = (\Phi(\mu_A), \Phi(v_A))$ is an intuitionistic fuzzy subsedmigroup of T. Again, let $x'y'a' \in T$ and $\alpha, \beta \in \Gamma$. Then, $\big(\Phi(\mu_A)\big)(x'\alpha a'\beta y') = sup_{\Phi(z)=x'\ \alpha a'\beta y'}\{\mu_A(z)\}$

$$\geq sup_{\Phi(x)=x',\Phi(a)=a',\Phi(y)=y'}\{\mu_A(x\alpha a\beta y)\}$$

$$\geq sup_{\Phi(a)=a'}\{\mu_A(a)\}$$

$$= \big(\Phi(\mu_A)\big)(a')$$

And $\big(\Phi(v_A)\big)(x'\alpha a'\beta y') = inf_{\Phi(z)=x'\ \alpha a'\beta y'}\{v_A(z)\}$

$$\leq inf_{\Phi(x)=x',\Phi(a)=a',\Phi(y)=y'}\{v_A(x\alpha a\beta y)\}$$

$$\leq inf_{\Phi(a)=a'}\{v_A(a)\}$$

$$= \big(\Phi(v_A)\big)(a').$$

Therefore, $\Phi(A) = (\Phi(\mu_A), \Phi(v_A))$ is an intuitionistic fuzzy interior ideal of T. ■

Proposition 3.2.3: Let S be a Γ-semigroup and $A = \langle \mu_A, v_A \rangle$ be an intuitionistic fuzzy ideal of S. Then, A is an intuitionistic fuzzy interior ideal of S.

Proof :

Let $A = \langle \mu_A, v_A \rangle$ be an intuitionistic fuzzy ideal of S. Let $x, y \in S$ and $\gamma \in \Gamma$. Then, $\mu_A(x\gamma y) \geq \mu_A(x)$ and $\mu_A(x\gamma y) \geq \mu_A(y)$, which implies that

$\mu_A(x\gamma y) \geq \min\{\mu_A(x), \mu_A(y)\}$. Again, we have $v_A(x\gamma y) \leq v_A(x)$ and $v_A(x\gamma y) \leq v_A(y)$, which implies $v_A(x\gamma y) \leq \max\{v_A(x), v_A(y)\}$. Hence, $A = \langle \mu_A, v_A \rangle$ is an intuitionistic fuzzy subsemigroup of S.

Now, let $x, a, y \in S$ and $\alpha, \beta \in \Gamma$. Then, $\mu_A(x\alpha a\beta y) = \mu_A(x\alpha(a\beta y)) \geq \mu_A(a\beta y) \geq \mu_A(a)$ and $v_A(x\alpha a\beta y) = v_A(x\alpha(a\beta y)) \leq v_A(a\beta y) \leq v_A(a)$.

Hence, $A = \langle \mu_A, v_A \rangle$ is an intuitionistic fuzzy interior ideal of S. ■

Proposition 3.2.4:

Let S be a regular Γ-semigroup and $A = \langle \mu_A, v_A \rangle$ be an intuitionistic fuzzy interior ideal of S. Then, A is an intuitionistic fuzzy ideal of S.

Proof :

Let $x, y \in S, \gamma \in \Gamma$. Since S is regular, for any $x \in S$ there exist $a \in S$ and $\alpha, \beta \in \Gamma$ such that $x = x\alpha a\beta x$. Then, $\mu_A(x\gamma y) = \mu_A(x\alpha a\beta x\gamma y) \geq \mu_A(x)$ and $v_A(x\gamma y) = v_A(x\alpha a\beta x\gamma y) \leq v_A(x)$. So $A = \langle \mu_A, v_A \rangle$ is an intuitionistic fuzzy right ideal of S.

Similarly, we can prove that $A = \langle \mu_A, v_A \rangle$ is an intuitionistic fuzzy left ideal of S. Hence, $A = \langle \mu_A, v_A \rangle$ is an intuitionistic fuzzy ideal of S.

Remark 3.2.1:

From the above two propositions it is clear that in regular Γ-semigroups the concept of intuitionistic fuzzy ideals and intuitionistic fuzzy interior ideals coincide.

Definition 3.2.6:

An interior ideal M of a Γ-semigroup S is called a characteristic interior ideal of S if $f(M) = M$ for all $f \in Aut(S)$.

Example3.2.4

Let S be the set of all 2×3 matrices and Γ be the set of all 3×2 matrices over the ring of integers. Then, S is a Γ-semigroup with respect to usual matrix multiplication.

Let us define $f : S \rightarrow S$ by $f(A_{2,3}) = 2A_{2,3}$. Then, $f \in Aut(S)$. Let $A = \langle \mu_A, v_A \rangle$ be an intuitionistic fuzzy subset of S defined as follows:

$$\mu_A(x) = \begin{cases} 0.3 & if \ \ x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ 0.2 & otherwise \end{cases}$$

And

$$v_A(x) = \begin{cases} 0.6 & if \ \ x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ 0.8 & otherwise \end{cases}$$

Then, $A = \langle \mu_A, v_A \rangle$ is an intuitionistic fuzzy characteristic interior ideal of S.

Definition 3.2.7:

An intuitionistic fuzzy interior ideal $A = \langle \mu_A, v_A \rangle$ of a Γ-semigroup S is called an intuitionistic fuzzy characteristic interior ideal of S if $\mu_A(f(x)) = \mu_A(x)$ and $v_A(f(x)) = v_A(x)\ \forall\ x \in S$ and $\forall\ x \in Aut(S)$.

Theorem 3.2.4:

If $A = \langle \mu_A, v_A \rangle$ is an intuitionistic fuzzy characteristic interior ideal of S then the upper and lower level cuts $U(\mu_A; t)$ and $L(v_A; t)$ are characteristic interior ideals of S for every $t \in Im(\mu_A) \cap Im(v_A)$.

Proof :

Let $t \in Im(\mu_A) \cap Im(v_A)$. Then, $U(\mu_A; t)$ and $L(v_A; t)$ are interior ideals of S. Let $f \in Aut(S)$ and $x \in U(\mu_A; t)$. Then, $\mu_A(x) \geq t$ and so $\mu_A(f(x)) \geq t$.

Hence, $f(x) \in U(\mu_A; t)$. This implies that $f(U(\mu_A; t) \subset U(\mu_A; t)$. Again, let $x \in U(\mu_A; t)$ and $y \in S$ such that $f(y) = x$. Then, $\mu_A(y) = \mu_A\big(f(y)\big) = \mu_A(x) \geq t$, so $y \in U(\mu_A; t)$. Consequently, $f(y) \in f(U(\mu_A; t))$, whence $x \in f(U(\mu_A; t)$. Hence, $U(\mu_A; t) \subset f(U(\mu_A; t))$. So, we have $U(\mu_A; t) = f(U(\mu_A; t))$.

Hence, $U(\mu_A; t)$ is a characteristic interior ideal of S. By using similar argument we can show that $L(v_A; t)$ is characteristic interior ideal of S.

■

Theorem 3.2.5:

If $A = \langle \mu_A, v_A \rangle$ is an intuitionistic fuzzy subset of S such that the non-empty sets $U(\mu_A; t)$ and $L(v_A; t)$ are characteristic interior ideals of S for all $t \in [0, 1]$. Then, $A = \langle \mu_A, v_A \rangle$ is an intuitionistic fuzzy characteristic interior ideal of S.

Proof :

Let $U(\mu_A;t)$ and $L(v_A;t)$ are characteristic interior ideals of S for all $t \in [0,1]$. Let $f \in Aut(S)$, $x \in S$ and $\mu_A(x) \coloneqq t_0$ and $v_A(x) \coloneqq t_1$. Then, $x \in U(\mu_A;t_0)$ and $x \in L(v_A;t_1)$. Since, by hypothesis, $U(\mu_A;t_0) = f(U(\mu_A;t_0))$ and $L(v_A;t_1) = f(L(v_A;t_1))$, so we see that $f(x) \in U(\mu_A;t_0)$ and $f(x) \in L(v_A;t_1)$. Hence, $\mu_A(f(x)) \geq t_0$ and $v_A(f(x)) \leq t_1$. Let $t_2 = \mu_A(f(x))$ and $t_3 = v_A(f(x))$. Then, $t_2 \geq t_0$ and $t_3 \leq t_1$ and $f(x) \in U(\mu_A;t_2) = f(U(\mu_A;t_2))$, $f(x) \in L(v_A;t_3) = f\big(L(v_A;t_3)\big)$.

Since f is one-one, we have $x \in U(\mu_A;t_2)$ and $x \in L(v_A;t_3)$. This implies that $\mu_A(x) \geq t_2$ and $v_A(x) \leq t_3$. Hence, $t_0 \geq t_2$ and $t_1 \leq t_3$. Thus, we obtain $\mu_A\big(f(x)\big) = \mu_A(x)$ and $v_A\big(f(x)\big) = v_A(x)$. Hence, $A = (\mu_A, v_A)$ is an intuitionistic fuzzy characteristic interior ideal of S. ■

Corollary 3.2.4:

Let χ_I be the characteristic function of a characteristic interior ideal I of S. Then, $I = (\chi_I, \chi_I^c)$ is an intuitionistic fuzzy characteristic interior ideal of S.

Theorem 3.2.6:

A Γ-semigroup is simple if and if only if every ituitionistic fuzzy interior ideal of S is constant.

Proof :

Let S be a simple Γ-semigroup and $A = \langle \mu_A, v_A \rangle$ be an intuitionistic fuzzy interior ideal of S. Let $a, b \in S$. Since S is simple, each for $< a >$ and $< b >$, the principal ideals generated by a and b , respectively, is equal to S.

Hence, there exist $x, y, u, v \in S$ and $\alpha, \beta, \gamma, \delta \in \Gamma$ such that $a = x\alpha b\beta y$ and $b = u\gamma a\delta v$.

Since $A = \langle \mu_A, v_A \rangle$ is an intuitionistic fuzzy interior ideal of S, so we have $\mu_A(x\alpha b\beta y) \geq \mu_A(b), \mu_A(u\gamma a\delta v) \geq \mu_A(a)$ and $v_A(x\alpha b\beta y) \leq v_A(b), v_A(u\gamma a\delta v) \leq v_A(a)$. Thus, we see that $\mu_A(a) \geq \mu_A(b), \mu_A(b) \geq \mu_A(a)$ and $v_A(a) \leq v_A(b), v_A(b) \leq v_A(a)$. Hence $\mu_A(a) = \mu_A(b)$ and $v_A(a) = v_A(b)$. Consequently, $A = \langle \mu_A, v_A \rangle$ is constant.

Conversely, suppose that every intuitionistic fuzzy interior ideal of S is constant. Let $A = \langle \mu_A, v_A \rangle$ be an intuitionistic fuzzy ideal of S. Then, by Proposition 2.3, $A = \langle \mu_A v_A \rangle$ is an intuitionistic fuzzy interior ideal of S. Hence, by hypothesis $A = \langle \mu_A, v_A \rangle$ is constant. Thus, S is intuitionistic fuzzy simple. Therefore, by lemma 2.3, S is a simple Γ-semigroup.

■